Drinfeld Displays

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Abstract

In this paper we develop the theory of Drinfeld displays. This is the equal characteristic analog of Zink's theory of displays. We study the category of Drinfeld displays, and prove that it is equivalent to the category of z-divisible local Anderson modules, which are the equal characteristic analogs of p-divisible groups. Some of other main results are faithfully flat descent and rigidity. We will also develop and study ramified Dieudonné theory in the context of z-divisible local Anderson modules.

Contents

0 Introduction

In this paper, we develop the theory of *Drinfeld displays*. Before giving the details, let us put their definition in perspective. Let $\mathcal{A}$ be an Abelian scheme of dimension g over a base scheme S. The p^n-torsion closed subgroups $\mathcal{A}[p^n]$,

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for varying n, form an inductive system of finite flat group schemes, and their limit is a p-divisible group over S, of height $2g$ and dimension g, denoted by $\mathcal{A}[p^\infty]$. This p-divisible group plays a crucial point in the arithmetic study of Abelian schemes. In fact p-divisible groups in themselves are of utmost importance (e.g., they are one of the key ingredients in the proof of local Langlands correspondence for p-adic fields). A tool in study and classification of p-divisible groups is their Dieudonné module over perfect fields of characteristic p, or more generally (arbitrary base rings), their display, as developed by Zink in [16].

In the equal characteristic case, the analog of Abelian schemes are Drinfeld-Anderson A-motives (and the analog of elliptic curves are Drinfeld modules). Let Φ be an Anderson A-module of dimension d and rank h over a $\mathbb{F}_q[z]$-scheme S. Assume that the image ζ of z in $\mathcal{O}_S$ is nilpotent. One can show that the z^n-torsion closed subgroups $\Phi[z^n]$, for varying n, form an inductive system of finite flat group schemes and their limit, denoted by $\Phi[z^\infty]$, is a *z-divisible local Anderson module* of height h and dimension d. These are the equal characteristic analogs of p-divisible groups, and are fppf sheaves of $\mathbb{F}_q[\![z]\!]$-modules G that satisfy:

(a) G is z-torsion,

(b) G is z-divisible,

(c) $\forall n$ the subsheaf $G[z^n]$ is representable by a finite flat $\mathbb{F}_q$-module scheme that is strict in the sense of Faltings and

(d) the action of $(z-\zeta)$ on the tangent space of G is Zariski locally nilpotent

In this paper, we define and develop the equal characteristic analogs of displays and call them *Drinfeld displays.* A Drinfeld display over a $\mathbb{F}_q[\![z]\!]$-scheme S is a pair $\mathcal{P} = (P, V)$, where

- P is sheaf of $\mathcal{O}_S[\![z]\!]$-modules over S that is, Zariski locally on S, a free $\mathcal{O}_S[\![z]\!]$-module and
- $V : P \to \sigma^* P$ is a an $\mathcal{O}_S[\![z]\!]$-linear map, such that $V[\frac{1}{z-\zeta}]$ is an isomorphism.

One of the main result of this paper is that the category of Drinfeld displays is equivalent to the category of z-divisible local Anderson modules. We prove this by first developing equivalences of categories for various kinds of finite flat $\mathbb{F}_q$-schemes (finite φ-modules, finite z-modules and finite v-modules) and then use a limit argument.

We also show that Drinfeld displays satisfy faithfully flat descent. This result is quite useful, as it readily implies faithfully flat descent for z-divisible local Anderson modules.

Another result shown about Drinfeld displays, is the rigidity in the sense of p-divisible groups. This is essentially saying that up to isogeny, the restriction of homomorphisms of Drinfeld displays over a scheme to a closed subscheme defined by a locally nilpotent ideal is an isomorphism. This shows that Drinfeld displays (and so z-divisible local Anderson modules) have crystalline nature.

Finally, when the base scheme is a perfect field of characteristic p, we also develop a Dieudonné theory, called *ramified Dieodonné theory*, that is more adequate in this setting.

In a sequel paper, we will further develop the theory of Drinfeld displays (their multilinear theory) and use it to show the existence of tensor constructions of z-divisible local Anderson modules.

I should note that a theory similar to our theory of Drinfeld display has been independently developed in [8], where they prove that the category of z-divisible local Anderson modules is *anti-equivalent* to the category of the so-called *effective local shtukas*. However, one advantage of Drinfeld displays is that their category is *equivalent* to the category of z-divisible local Anderson modules, which is indispensable in some applications, and in particular in those related to the multilinear theory of local Anderson modules.

Notations 0.1.

- p is a prime number different from 2, $q = p^f$ is a power of p and $\mathbb{F}_q$ is the finite field with q elements.
- $\mathcal{O} = \mathbb{F}_q[\![z]\!]$ is the ring of formal power series in variable z and coefficients in $\mathbb{F}_q$.
- S is a fixed $\mathcal{O}$-scheme.
- Denote by ζ the image of z in $\mathcal{O}_S$ that we assume is locally nilpotent on S.
- If R is a ring and r is an element of R, we denote by R/r the quotient ring R/rR.
- For a ring R, we denote by $\mathfrak{Mod}_R$ the category of R-modules.
- For a scheme T, we denote by $\mathfrak{Sch}_T$ the category of schemes over T.
- Let X be a scheme over a base scheme S. We identify X with the sheaf $\mathrm{Mor}_S(\_, X)$ on fppf site of S.
- Let X be a scheme over a base scheme S and $f : T \to S$ a morphism. We denote by X_T the fiber product $X \times_S T$. If $\mathcal{F}$ is a sheaf on a Grothendieck site over S,

we denote by $f^*\mathcal{F}$ the pullback of $\mathcal{F}$ along f. So f^*X and X_T are identified as sheaves.

- Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on a Grothendieck site. We denote by $\underline{\mathrm{Mor}}(\mathcal{F}, \mathcal{G})$ the sheaf of morphisms from $\mathcal{F}$ to $\mathcal{G}$. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves of abelian groups, we denote by $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ the sheaf of homomorphisms.
- In this paper, all group schemes are assumed to be commutative.
- The order of a finite group scheme G is denoted by $|G|$.
- In this paper, unless otherwise stated, all schemes are defined over $\mathbb{F}_q$.
- For a scheme X (defined over $\mathbb{F}_q$) we denote by $\sigma : X \to X$ the f^{th}-power of the absolute Frobenius, and continue to call it the absolute Frobenius of X. This is the $\mathbb{F}_q$-morphism which is the identity on the underlying topological space of X and is the $\mathbb{F}_q$-linear morphism $x \mapsto x^q$ on $\mathcal{O}_X$.
- For a group scheme G defined over the base scheme S, we denote by $F_G : G \to G^{(q)}$ the relative $\mathbb{F}_q$-linear Frobenius morphism of G over S (which is the f^{th}-iterate of the usual relative Frobenius $G \to G^{(p)}$).

1 z-Divisible Local Anderson Modules

Recall that in this paper all schemes are defined over $\mathbb{F}_q$.

Definition 1.1. Let R be a ring and X a scheme. An *R-module scheme* over X is a representable fppf-sheaf of R-modules over X. When X is an R-scheme and for all $r \in R$, the action of r induces the scalar multiplication of r on the tangent space of G (via the morphism $R \to \mathcal{O}_X$), we call G a *strict R-module scheme*. *Morphisms* of R-module schemes over X are morphisms of fppf-sheaves of R-modules over X. ▲

Example 1.2. The vector group scheme $\mathbb{G}_{a,X}^N$ ($N \geq 1$) over X has a natural structure of a strict $\Gamma(X, \mathcal{O}_X)$-module scheme over X. ■

Definition 1.3. A *finite φ-module*, G, over a scheme X is a finite flat strict $\mathbb{F}_q$-module scheme (or $\mathbb{F}_q$-vector space) over X such that there exists an embedding (of $\mathbb{F}_q$-module schemes) of G into a vector group scheme over X of some rank N, i.e., a group scheme that is Zariski locally isomorphic to $\mathbb{G}_{a,X}^N$. ▲

Remark 1.4. Finite φ-modules are *group schemes with strict $\mathbb{F}_q$-action* in Faltings' terminology (cf. [5]). ♢

Example 1.5. (i) Let X be a scheme. The kernel of the n-th power of the Frobenius $F_{\mathbb{G}_a}^n : \mathbb{G}_{a,X} \to \mathbb{G}_{a,X}$, denoted by α_{q^n}, is a finite φ-module over X.

(ii) Let M be a finite dimensional $\mathbb{F}_q$-vector space, then the constant $\mathbb{F}_q$-module scheme $\underline{M}$ is a finite φ-module. ■

Remark 1.6. Let G be a finite φ-module over a scheme X. Then the Zariski sheaf $\mathcal{E}_G := \underline{\mathrm{Hom}}_{\mathbb{F}_q,X}(G, \mathbb{G}_a)$ on X of $\mathbb{F}_q$-linear morphisms is a locally free $\mathcal{O}_X$-module of rank $\log_q |\mathcal{O}_G|$, and it generates the $\mathcal{O}_X$-algebra $\mathcal{O}_G$ (cf. [14] or [13] §2.4). It follows that G can be embedded canonically into the vector group scheme $\mathbb{V}_G := \mathbf{Spec}(\mathrm{Sym}_{\mathcal{O}_X} \mathcal{E}_G)$ as an $\mathbb{F}_q$-module scheme. We call $\mathbb{V}_G$ the *ambient space* of G. $\diamondsuit$

Definition 1.7. (i) Let G be an $\mathcal{O}$-module scheme over S killed by z^n and whose underlying $\mathbb{F}_q$-module scheme is a finite φ-module. If the $\mathcal{O}/z^n \otimes_{\mathbb{F}_q} \mathcal{O}_S$-module $\mathcal{E}_G$ is locally free, G is called a *finite z-module of degree n* over S.

(ii) Morphisms of finite z-modules are morphisms of the underlying $\mathcal{O}$-module schemes.

$\blacktriangle$

Definition 1.8. (i) A *finite v-module* is a pair (G, V_G), where G is a finite z-module, and $V_G : \mathbb{V}_G^{(q)} \to \mathbb{V}_G$ (see Remark 1.6) is an $\mathbb{F}_q$-linear morphism such that the restriction of the composition $V_G \circ F_{\mathbb{V}_G}$ to G is equal to the multiplication by $(z - \zeta)^N$ for some $N \geq 1$, called *the order of nilpotence* of G. If $N = 1$, the pair (G, V_G) or G is called *strict*.

(ii) A morphism $u : (G, V_G) \to (H, V_H)$ of finite v-modules is a morphism of finite z-modules such that the following induced diagram is commutative:

$$\begin{array}{ccc} \mathbb{V}_G^{(q)} & \xrightarrow{\mathbb{V}_u^{(q)}} & \mathbb{V}_H^{(q)} \\ \Big\downarrow{\scriptstyle V_G} & & \Big\downarrow{\scriptstyle V_H} \\ \mathbb{V}_G & \xrightarrow[\mathbb{V}_u]{} & \mathbb{V}_H \end{array}$$

We denote by $\mathrm{Hom}_V(G, H)$ the group of such morphisms. $\blacktriangle$

Remark 1.9. If ζ is zero on S, then V_G restricts to a morphism $G^{(q)} \to G$ of finite φ-modules (see Remark 3.36). $\diamondsuit$

Definition 1.10. A short exact sequence $0 \to G' \to G \to G'' \to 0$ of finite φ-modules over S is a short exact sequence of the underlying fppf sheaves. $\blacktriangle$

Definition 1.11. (i) A *z-divisible local Anderson module* over S (see Notations 0.1) is an fppf sheaf of $\mathcal{O}$-modules, G, over S satisfying the following properties:

- G is z-torsion, i.e., $G = \varinjlim G_n$, where G_n is the kernel of multiplication by z^n
- G is z-divisible, i.e., the multiplication by z is an epimorphism of G

- G_n seen as sheaves of $\mathbb{F}_q$-modules are representable by finite φ-modules over S
- the action of $(z-\zeta)$ on the tangent space $\mathcal{L}ie\, G := \varinjlim \mathcal{L}ie\, G_n$ is Zariski locally nilpotent.

(ii) The order of G_1 is of the form q^h, where $h : S \to \mathbb{Q}_{\geq 0}$ is a locally constant function, called the *height* of G. If G_n are strict $\mathcal{O}$-module schemes, or equivalently, if $(z-\zeta)$ is zero on the tangent space of G, we call G a *z-divisible module* or a *strict* z-divisible local Anderson module.

(ii) G is said to be *infinitesimal (respectively étale)* if all G_n are infinitesimal (respectively étale) group schemes over S.

▲

Remark 1.12. Let G be a z-divisible local Anderson module.

1) We will later show that G_n are finite z-modules (see Remark 3.54 2)).
2) For every n, m, we have a short exact sequence:

$$0 \to G_n \hookrightarrow G_{n+m} \xrightarrow{z^n} G_m \to 0.$$

3) One can show that the height of a z-divisible local Anderson module has integer values (see e.g. [10], Theorem B.14).
4) The order of G_n is equal to q^{nh}. ♢

Remark 1.13. Assume that we have an inductive system $G_1 \hookrightarrow G_2 \hookrightarrow \dots$ of closed immersions of finite flat $\mathcal{O}$-module schemes over S satisfying the following properties:

- G_n are finite φ-modules over S
- there exists a locally constant function $h : S \to \mathbb{Q}_{\geq 0}$ such that for all n, the order of G_n, as a locally constant function on S, is q^{nh}
- for all n, the sequence $0 \to G_n \to G_{n+1} \xrightarrow{z^n} G_{n+1}$ is exact
- the action of $(z-\zeta)$ on the tangent space $\mathcal{L}ie\, G := \varinjlim_n \mathcal{L}ie\, G_n$ is locally nilpotent.

then the limit $G := \varinjlim_n G_n$ is a z-divisible local Anderson module over S of height h, and for every n, we have $G[z^n] \cong G_n$. ♢

Remark 1.14. One can show that $\mathcal{L}ie\, G$ is a locally free $\mathcal{O}_S$-module of finite rank (cf. [8] Lemma 7.2(c)) and we define the *dimension* of G to be this finite rank. ♢

Remark 1.15. Let G be a (strict) z-divisible local Anderson module. We will later show (Corollary 3.58) that each G_n has a natural structure of a (strict) finite v-module. $\Diamond$

Example 1.16. Let C be a smooth projective geometrically connected curve over $\mathbb{F}_q$ and ∞ a closed point of C. Let A be the ring of functions on C which are regular outside ∞, i.e., $A = \Gamma(C \smallsetminus \{\infty\}, \mathcal{O}_C)$. Assume that S is an A-scheme and let $\mathcal{A}$ be an Anderson A-module of dimension d and rank h over S. Let $z \in A$ be a prime element, then $\mathcal{A}[z^\infty] := \varinjlim_n \mathcal{A}[z^n]$ is a z-divisible local Anderson module over S, of dimension d and height h. $\blacksquare$

Example 1.17. Let F be a formal $\mathcal{O}$-module over S of dimension d and height h, then $F[z^\infty] := \varinjlim_n F[z^n]$ is a (strict) z-divisible module over S of dimension d and height h. $\blacksquare$

Notations 1.18. Let X be a scheme. Denote by $\underline{\alpha}_q$ the inverse limit $\varprojlim_n \alpha_{q^n}$, where transition morphisms $\alpha_{q^{n+1}} \twoheadrightarrow \alpha_{q^n}$ are induced by Frobenius.

2 Ramified Dieudonné theory

In this section, k is a perfect field, which is an $\mathcal{O}$-algebra and we assume that z is in the kernel of $\mathcal{O} \to k$. Denote by σ the Frobenius automorphism of k (sending x to x^p). We would like to have a refinement of the Dieudonné theory for finite φ-modules and z-divisible local Anderson modules, which reflects the presence of $\mathcal{O}$ as a ring acting on these objects.

Lemma 2.1. *Let E/F be a finite Galois extension of fields and let K/E be a field extension. Let $\mathcal{F}$ and $\mathcal{G}$ be respectively a sheaf of E-vector spaces and K-vector spaces over the fppf site of a scheme X. Then the decomposition $E \otimes_F K = \prod_{\mathrm{Gal}(E/F)} K$ induces a decomposition of the $E \otimes_F K$-module*

$$\mathrm{Hom}_F(\mathcal{F}, \mathcal{G}) = \prod_{\mathrm{Gal}(E/F)} H_\sigma,$$

where H_σ is the subgroup $e_\sigma \cdot \mathrm{Hom}_F(\mathcal{F}, \mathcal{G})$ of $\mathrm{Hom}_F(\mathcal{F}, \mathcal{G})$ and e_σ is the element $(0, \dots, 0, 1, 0, \dots, 0)$ with 1 at place σ. Moreover, we have $H_{\mathrm{Id}} = \mathrm{Hom}_E(\mathcal{F}, \mathcal{G})$.

Proof. Standard. $\square$

Definition 2.2. Let G be a finite φ-module over k. The *ramified Dieudonné module* of G is the k-vector space $\mathrm{Hom}_{\mathbb{F}_q}(G, \mathbb{G}_a)^\vee$, where k acts through its canonical action on $\mathbb{G}_a$, and $(\_)^\vee$ denotes the k-dual. We denote the ramified Dieudonné module of G by $\mathbb{H}(G)$. $\blacktriangle$

Remark 2.3. 1) As in the "classical" Dieudonné theory, the f^{th} iterate of the Frobenius, $F_G^f : G \to G^{(q)}$, induces a σ^{-f}-linear endomorphism $V_f : \mathbb{H}(G) \to \mathbb{H}(G)$.

2) Since by definition G embeds, as a group scheme, in a power of the additive group $\mathbb{G}_a$ (Remark 1.6), the Verschiebung of G is trivial. It follows that the (covariant) Dieudonné module $\mathbb{D}(G)$ is a $k\{V\}$-module, where $k\{V\}$ is the non-commutative polynomial ring with $V\sigma(x) = xV$ for all $x \in k$ (note that the operator V corresponds to Frobenius on G).

3) Since the Verschiebung morphism of G is trivial, it is a unipotent group and therefore, its Dieudonné module is canonically isomorphic to the k-vector space $\mathrm{Hom}(G, \mathbb{G}_a)^\vee$. If furthermore G is local (i.e., infinitesimal), it is canonically isomorphic to $\mathrm{Hom}(\underline{\alpha}_q, G)$ of k-morphisms from the inverse limit $\underline{\alpha}_q$ (defined in Notation 1.18) to G (cf. Remark 1.13 of [10]).

4) When G is local, one can show that $\mathbb{H}(G)$ is canonically isomorphic to $\mathrm{Hom}_{\mathbb{F}_q}(\underline{\alpha}_q, G)$, where again, k acts through its canonical action on the limit $\underline{\alpha}_q$. ♢

Proposition 2.4. *Let G be a finite φ-module over k. There is a decomposition*

$$\mathbb{D}(G) \cong \mathbb{H}(G) \times V\mathbb{H}(G) \times \cdots \times V^{f-1}\mathbb{H}(G),$$

where V is the Verschiebung of the Dieudonné module, $\mathbb{D}(G)$, of G.

Proof. By Lemma 2.1, we have a decomposition

$$\mathbb{D}(G) \cong H_0 \times H_1 \times \cdots \times H_{f-1}$$

with $H_0 = \mathbb{H}(G)$. The Verschiebung V induces σ^{-1}-linear maps $H_i \to H_{i+1}$ for all $i \in \mathbb{Z}/f\mathbb{Z}$. The Lie algebra of G is canonically isomorphic to cokernel of V, and so we have

$$\mathcal{L}ie\, G \cong \mathbb{D}(G)/V\mathbb{D}(G) \cong H_0/VH_{f-1} \times \cdots \times H_{f-1}/VH_{f-2}.$$

By assumption, the action of $\mathbb{F}_q$ on the Lie algebra of G is scalar, but on the above product, the action of $\mathbb{F}_q$ is scalar only on the first factor (on other factors the action is given by a non-trivial Frobenius twist), which means that the other factors are trivial: for $i = 1, \ldots, f-1$, we have $H_i = VH_{i-1}$. The proposition is now proved. □

Remark 2.5. Note that by definition, we have $V_f = V^f|_{\mathbb{H}(G)}$. ♢

Definition 2.6. (i) A *finite Dieudonné module over k with scalar $\mathbb{F}_q$-action* is an $\mathbb{F}_q \otimes_{\mathbb{F}_p} k\{V\}$-module which is a finite free $\mathbb{F}_q \otimes_{\mathbb{F}_p} k$-module and the action of $\mathbb{F}_q$ on the cokernel of V is scalar (i.e. for every $x \in \mathbb{F}_q$, the action of the element $x \otimes 1 - 1 \otimes x$ on the cokernel of V is the zero endormophism). Denote by $\mathfrak{D}_{k,\mathbb{F}_q}$ the category of finite Dieudonné modules over k with scalar $\mathbb{F}_q$-action.

(ii) A *ramified Dieudonné module over k* is a $k\{V_f\}$-module that has finite dimension as a k-vector space. Here $k\{V_f\}$ is the noncommutative polynomial ring in the variable V_f with $V_f \sigma^f(x) = xV_f$ for all $x \in k$. Denote by $\mathfrak{rD}_k$ the category of ramified Dieudonné modules over k.

(iii) Denote by $\mathbb{E}(k)$ and respectively $\mathbb{E}_f(k)$ the non-commutative rings $k\{V\}$ and $k\{V_f\}$. ▲

Construction 2.7. Let D be an object in $\mathfrak{D}_{k,\mathbb{F}_q}$. The arguments given in the proof of Proposition 2.4 give a decomposition of k-vector spaces

$$D \cong H_0 \times VH_0 \times \cdots \times V^{f-1}H_0.$$

Denote H_0 by $H(D)$ and call it the *ramified Dieudonné module of D*. This is indeed a ramified Dieudonné module over k with V_f acting via V^f. Note that since D is a free $\mathbb{F}_q \otimes_{\mathbb{F}_p} k$-module of finite rank, for all $i = 0. \ldots, f-1$, the summands H_i in the decomposition $D \cong H_0 \times \cdots \times H_{f-1}$ have the same dimension as k-vector spaces, and therefore, since $VH_{i-1} = H_i$ $(i = 1, \ldots, f-1)$, the Verschiebung induces σ^{-1}-linear isomorphisms $V : H_{i-1} \to H_i$ $(i = 1, \ldots, f-1)$.

Conversely, let H be a ramified Dieudonné module over k. Set $D(H) := H^{\oplus f}$, the product of f copies of the k-vector space H, and let V and $\mathbb{F}_q$ act on it as follows:

$$V \cdot (x_0, \ldots, x_{f-1}) := (V_f x_{f-1}, x_0, \ldots, x_{f-2}),$$
$$\forall r \in \mathbb{F}_q : r \cdot (x_0, \ldots, x_{f-1}) := (rx_0, \sigma(r)x_1, \ldots, \sigma^{f-1}(r)x_{f-1}).$$

It is now easy to see that $D(H)$ is a finite Dieudonné module over k with scalar $\mathbb{F}_q$-action. Call it the *finite Dieudonné module of H*. ▼

Proposition 2.8. *The functors $H : \mathfrak{D}_{k,\mathbb{F}_q} \to \mathfrak{rD}_k$ and $D : \mathfrak{rD}_k \to \mathfrak{D}_{k,\mathbb{F}_q}$ are quasi-inverses to each other.*

Proof. The proof is straightforward and is left to the reader. □

Corollary 2.9. *Let G be a finite φ-module over k. Then:*

(a) The ramified Dieudonné module $\mathbb{H}(G)$ is a ramified Dieudonné module over k.

(b) The Dieudonné module $\mathbb{D}(G)$ is a finite Dieudonné module over k with scalar $\mathbb{F}_q$-action.

Proof. Part (a) follows from the construction of $\mathbb{H}(G)$. Part (b) follows from Propositions 2.4 and 2.8. □

Remark 2.10. We will prove later (cf. Corollary 3.15) that the category of ramified Dieudonné modules over k is equivalent to the category of finite φ-modules over k. It follows from the above proposition that the latter is equivalent to the category $\mathfrak{D}_{k,\mathbb{F}_q}$ as well. ◊

Definition 2.11. Let G be a z-divisible local Anderson module over k. The *ramified Dieudonné module* of G is the inverse limit $\varprojlim_n \mathbb{H}(G_n)$ where the transition morphisms are induced by the projections $z. : G_{n+1} \twoheadrightarrow G_n$. We denote it by $\mathbb{H}(G)$. ▲

Proposition 2.12. *Let G be a z-divisible local Anderson module over k of height h. The ramified Dieudonné module $\mathbb{H}(G)$ is a free $k[\![z]\!]$-module of rank h.*

Proof. This is a special case of Proposition 3.47 (using the identification of Theorem 3.57 (f)). But let us give an elementary proof here:

Let us write $\mathbb{H}$ and respectively $\mathbb{H}_n$ for the ramified Dieudonné module of G and respectively G_n. Multiplication by z on G_n induces an action of z on $\mathbb{H}_n$, commuting with the $\mathbb{F}_q$-action, making it a module over $k \otimes_{\mathbb{F}_q} \mathbb{F}_q[z] \cong k[\![z]\!]$. Since G_n is killed by z^n, it is naturally a $k[z]/z^n$-module. This implies that $\mathbb{H}$, being the inverse limit of $\mathbb{H}_n$, has a natural structure of an $k[\![z]\!]$-module.

The ramified Dieudonné module $\mathbb{H}_n$ is finite over k. Let $d_1, \cdots, d_r$ be elements in $\mathbb{H}$ whose images in $\mathbb{H}_1$ is a basis over k and define a morphism $k[\![z]\!]^r \to \mathbb{H}$ by sending basis elements to d_i. This morphism induces morphisms

$$(k[\![z]\!]/(z^n))^r \to \mathbb{H}/z^n\mathbb{H} \cong \mathbb{H}_n$$

which are surjective (since modulo z they are surjective and using Nakayama's lemma) an so, being an inverse limit of surjective morphisms, $k[\![z]\!]^r \to \mathbb{H}$ is surjective (note that the transition morphisms are also surjective and so the Mittag-Leffler condition is satisfied). This implies that $\mathbb{H}$ is a finite module over $k[\![z]\!]$. The action of z on G is surjective and therefore its action on $\mathbb{H}$ is injective. It follows that $\mathbb{H}$ is a torsion-free $k[\![z]\!]$-module and hence is free of rank r over it, since $k[\![z]\!]$ is a principal ideal domain. It remains to show that $r = h$. The order of the finite group scheme G_1 is $q^h = p^{fh}$. And so its (classical)

Dieudonné module $\mathbb{D}(G_1)$ has dimension fh over k. The same arguments as in the proof of Proposition 2.4 show that we have a natural decomposition

$$\mathbb{D}(G) \cong \mathbb{H} \times V\mathbb{H} \times \cdots \times V^{f-1}\mathbb{H}$$

and since $V^i : \mathbb{H} \to V^i\mathbb{H}$ $(i = 0, \ldots, f-1)$ is injective (see e.g. [10] Lemma B.13), $\mathbb{H}$ and $V^i\mathbb{H}$ have the same rank over $k[\![z]\!]$, and so do $\mathbb{H}_1$ and $V^i\mathbb{H}_1$ over k. It now follows from Proposition 2.4 that $\mathbb{H}_1$ has dimension h and therefore $r = h$. □

3 Drinfeld Displays

In this section, we are going to define and study Drinfeld Displays and some related constructions.

Notations 3.1. Unless otherwise specified, by Frobenius of a scheme over $\mathbb{F}_q$ we mean the $\mathbb{F}_q$-linear Frobenius (corresponding to the map $x \to x^q$). Let S be a fixed $\mathcal{O}$-scheme. Denote by $\sigma_S : \mathcal{O}_S[\![z]\!] \to \mathcal{O}_S[\![z]\!]$ the endomorphism that fixes z and restricts to the Frobenius morphism σ on $\mathcal{O}_S$. When confusion is not likely, we write σ instead of σ_S.

3.1 Finite sheaves

Definition 3.2. (i) A *Ver-sheaf* over S is a pair $(\mathcal{D}, v)$, where $\mathcal{D}$ is a finite locally free $\mathcal{O}_S$-module and $v : \mathcal{D} \to \sigma^*\mathcal{D}$ is an $\mathcal{O}_S$-linear map. A *nilpotent Ver-sheaf* is a Ver-sheaf $(\mathcal{D}, v)$, where v is nilpotent, in the sense that there is some $n \geq 1$ such that the composition $\sigma^{n*}v \circ \cdots \circ \sigma^*v \circ v$ is zero. Morphisms of Ver-sheaves are morphisms of $\mathcal{O}_S$-modules commuting with v.

(ii) Let $(\mathcal{D}, v)$ be a Ver-sheaf over S with an $\mathcal{O}$-action and assume that it is killed by z^n. If the $\mathcal{O}/z^n \otimes_{\mathbb{F}_q} \mathcal{O}_S$-module $\mathcal{D}$ is Zariski locally free, then we call $(\mathcal{D}, v)$ a *z-sheaf of degree n*. Morphisms of z-sheaves are morphisms of the underlying Ver-sheaves, which commute with the $\mathcal{O}$-action.

(iii) A *Frob-sheaf* over S is a triple $(\mathcal{D}, v, \varphi)$, where $\mathcal{D}$ is an $\mathcal{O}_S[\![z]\!]$-module killed by a power of z, $(\mathcal{D}, v)$ is a Ver-sheaf over S and $\varphi : \sigma^*\mathcal{D} \to \mathcal{D}$ is an $\mathcal{O}_S$-linear map such that $\varphi \circ v = (z - \zeta)^N$ for some $N \geq 1$. The integer N is called the *the order of nilpotence*. If $N = 1$, $(\mathcal{D}, v, \varphi)$ is called *strict*.

(iv) A *finite $\mathbb{F}_q$-shtuka* over S is a pair $(\mathcal{E}, \varphi)$, where $\mathcal{E}$ is a finite locally free $\mathcal{O}_S$-module and $\varphi : \sigma^*\mathcal{E} \to \mathcal{E}$ is an $\mathcal{O}_S$-linear map[1]. Morphisms of finite $\mathbb{F}_q$-shtukas are morphisms of $\mathcal{O}_S$-modules commuting with φ.

▲

[1] In [4, 14] these are called φ-sheaves

Example 3.3. Let X be a scheme. Example 1.25, p.16 of [9] states that we have an isomorphism

$$\mathcal{O}_X^m \xrightarrow{\cong} \underline{\mathrm{Hom}}(\alpha_{p^m}, \mathbb{G}_a)$$
$$(x_1, \dots, x_m) \mapsto [a \mapsto \sum_{i=1}^{m} x_i a^{p^{i-1}}]$$

The same arguments show that the map

$$\mathcal{O}_X^m \xrightarrow{\cong} \underline{\mathrm{Hom}}_{\mathbb{F}_q}(\alpha_{q^m}, \mathbb{G}_a) = \mathcal{E}_{\alpha_{q^m}} \tag{3.4}$$
$$(x_1, \dots, x_m) \mapsto [a \mapsto \sum_{i=1}^{m} x_i a^{q^{i-1}}]$$

is an isomorphism as well. One can then easily check that the Frobenius φ of $\mathcal{E}_{\alpha_{q^m}}$, in the standard basis of $\mathcal{O}_X^m$, is given by the $m \times m$ matrix

$$\Phi_m := \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix} \tag{3.5}$$

■

Definition 3.6. Let G be a finite φ-module over S. Recall from Remark 1.6 that $\mathcal{E}_G = \underline{\mathrm{Hom}}_{\mathbb{F}_q.S}(G, \mathbb{G}_a)$ is a locally free sheaf.

(i) The finite $\mathbb{F}_q$-shtuka of G is defined to be $\mathcal{E}_G$, with $\varphi : \sigma^*\mathcal{E}_G \to \mathcal{E}_G$ induced by the Frobenius of G.

(ii) The Ver-sheaf of G is defined to be the Zariski sheaf $\mathcal{D}(G) := \mathcal{E}_G^\vee = \underline{\mathrm{Hom}}_{\mathcal{O}_S}(\mathcal{E}_G, \mathcal{O}_S)$. In other words, $\mathcal{D}(G)$ is canonically isomorphic to the Zariski sheaf of sections of $\mathbb{V}_G$ over S (see Remark 1.6). By functoriality, the Frobenius $F_G : G \to G^{(q)}$ induces an $\mathcal{O}_S$-linear morphism $v_G : \mathcal{D}(G) \to \sigma^*\mathcal{D}(G)$.

▲

Definition 3.7. Let (G, V_G) be a finite v-module over S (cf. Definition 1.8). The morphism $V_G : \mathbb{V}_G^{(q)} \to \mathbb{V}_G$ induces an $\mathcal{O}_S$-linear morphism $\mathcal{E}_G \to \mathcal{E}_G^{(q)}$, as $\mathcal{E}_G$ is the sheaf of sections of the vector bundle $\mathbb{V}_G$. By duality, we obtain an $\mathcal{O}_S$-linear morphism $\varphi_G : \sigma^*\mathcal{D}(G) \to \mathcal{D}(G)$, which satisfies the identity $\varphi_G \circ v_G = (z - \zeta)^N$, where N is the order of nilpotence of G. The triple $(\mathcal{D}(G), v_G, \varphi_G)$ is the Frob-sheaf of G. ▲

Remark 3.8. 1) One can show that if G is a finite φ-module, then its tangent space is canonically isomorphic to the $\mathcal{O}_S$-dual of $\mathrm{Coker}(\varphi : \sigma^*\mathcal{E}_G \to \mathcal{E}_G)$ (see [4], Proposition 2.1).

2) Although by definition $\mathcal{D}(G)$ and $\mathcal{E}(G)$ are Zariski sheaves, we can attach to them fppf sheaves in a canonical way: let $\mathcal{F}$ be a Zariski sheaf on a scheme X, then for any X-scheme $a : Y \to X$, we define $\mathcal{F}(Y)$ to be $\Gamma(Y, a^*\mathcal{F})$, i.e., the global section of the pullback of $\mathcal{F}$. We will still denote this fppf sheaf by $\mathcal{F}$. If $\mathcal{F}$ is locally free as a Zarsiki sheaf (of rank n), the corresponding fppf sheaf is locally free (of rank n) as well.

3) Assume that the base scheme S is the spectrum of a perfect field k and let $(\mathcal{D}, v)$ be a Ver-sheaf on S. Then we can identify the Zariski sheaf $\mathcal{D}$ with its global sections and v induces a σ^{-1}-linear endomorphism on $\mathcal{D}$. Therefore, $\mathcal{D}$ is canonically a ramified Dieudonné module over k. Under the above identification, the Ver-sheaf of a finite φ-module G is canonically isomorphic to its ramified Dieudonné module $\mathbb{H}(G)$. ♢

Proposition 3.9. *The functor $G \mapsto (\mathcal{E}_G, \varphi_G)$ is an anti-equivalence of categories from the category of finite φ-modules over S to the category of finite $\mathbb{F}_q$-shtukas over S. Under this equivalence, G is infinitesimal (respectively étale) if and only if φ_G is nilpotent (respectively an isomorphism). Moreover, we have* $\mathrm{rank}_{\mathcal{O}_S} \mathcal{E}_G = \log_q |G|$.

Proof. We refer to [14], Proposition (1.7) or [8] Theorem 5.2 for the proof. Here we only give the construction of the quasi-inverse functor

$$(\mathcal{E}, \varphi) \mapsto Gr(\mathcal{E}, \varphi)$$

Let $(\mathcal{E}, \varphi)$ be a finite $\mathbb{F}_q$-shtuka over S and $E := \mathbb{V}(\mathcal{E}) = \mathbf{Spec}(\mathrm{Sym}_{\mathcal{O}} \mathcal{E})$ the associated vector bundle over S. By functoriality, $\varphi : \sigma^*\mathcal{E} \to \mathcal{E}$ induces a morphism $\Phi : E \to E^{(q)}$ (note that this is different from the relative Frobenius F_E of the S-scheme E). Define

$$Gr(\mathcal{E}, \varphi) := \mathrm{Ker}\left(\Phi - F_E : E \to E^{(q)}\right)$$

This is the subgroup scheme of $E = \mathbf{Spec}(\mathrm{Sym}_{\mathcal{O}} \mathcal{E})$ given by the ideal generated by $(\varphi - \sigma)(\mathcal{E})$, where $\sigma : \mathrm{Sym}_{\mathcal{O}} \mathcal{E}^{(q)} \to \mathrm{Sym}_{\mathcal{O}} \mathcal{E}$ is the relative Frobenius ring homomorphism of $\mathrm{Sym}_{\mathcal{O}} \mathcal{E}$. □

Remark 3.10. It follows from the construction of the functor Gr that if $(\mathcal{E}, \varphi)$ is a finite $\mathbb{F}_q$-shtuka over S, then for any S-scheme T, the $\mathbb{F}_q$-vector space $Gr(\mathcal{E}, \varphi)(T)$ is the subspace of $\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{E}, \mathcal{O}_T)$ consisting of $\mathcal{O}_S$-linear morphisms $u : \mathcal{E} \to \mathcal{O}_T$ with the property that for all sections $x \in \mathcal{E}$, we have

$$u(x)^q = u\big(\varphi(x \otimes 1)\big) \tag{3.11}$$

◊

Proposition 3.12. *The functor $G \mapsto (\mathcal{D}(G), v_G)$ is an equivalence of categories from the category of finite φ-modules over S to the category of Ver-sheaves over S and we have $\operatorname{rank}_{\mathcal{O}_S} \mathcal{D}(G) = \log_q |G|$. Under this equivalence, G is infinitesimal (respectively étale) if and only if v_G is nilpotent (respectively an isomorphism). Moreover, the quasi-inverse of this functor, denoted by $\mathcal{G}$ is given as follows: if $(\mathcal{D}, v)$ is a Ver-sheaf over S, then for any S-scheme T, we have*

$$\mathcal{G}(\mathcal{D}, v)(T) = \{x \in \mathcal{D}(T) \mid v(x) = x \otimes 1\} \tag{3.13}$$

In other words, $\mathcal{G}(\mathcal{D}, v)$ is the equalizer of the $\mathbb{F}_q$-linear maps

$$\mathcal{D} \underset{-\otimes 1}{\overset{v}{\rightrightarrows}} \sigma^*\mathcal{D} \tag{3.14}$$

Proof. The first statement follows at once from the previous proposition, by observing that the duality $\mathcal{E} \mapsto \mathcal{E}^\vee$ induces an anti-equivalence of categories from the category of locally free $\mathcal{O}_S$-modules of a fixed rank to itself, and it restricts to an anti-equivalence from the category of finite $\mathbb{F}_q$-shtukas to the category of Ver-sheaves.

For the second statement, note that the quasi-inverse of the functor $\mathcal{D}$ is the composite of the functors $(\_)^\vee$ and Gr. The equality (3.13) now follows from the explicit description (3.11) of the functor Gr given in Remark 3.10. □

Corollary 3.15. *The functor $G \mapsto \mathbb{H}(G)$ is an equivalence of categories from the category of finite φ-modules over a perfect field k to the category of ramified Dieudonné modules over k.*

Proof. This follows from the above Proposition 3.12 and Remark 3.8. □

Remark 3.16. Let G be a finite φ-module over S of order q^n. The associated Ver-sheaf $\mathcal{D}(G)$ is locally free of rank n. Now, assume further that $\mathcal{D}(G)$ is free. By the previous proposition, we can canonically identify G with $\mathcal{G}(\mathcal{D}(G))$ and so for any S-scheme T, every element of $G(T)$ is given by an n-tuple in $\mathcal{O}_T(T)$. In particular, the identity element $\mathrm{Id}_G \in G(G)$ is given by n elements in $\mathcal{O}_G(G)$. It is easy to see that these elements are linearly independent over $\mathcal{O}_S(S)$. In fact, viewing $\mathcal{E}_G(S) = \operatorname{Hom}_{\mathbb{F}_q,S}(G, \mathbb{G}_a)$ as a subset of $\mathcal{O}_G(G)$, these elements form a basis of $\mathcal{E}_G(S)$ over $\mathcal{O}_S(S)$. ◊

Remark 3.17. Let G be a finite z-module of degree n over S. By definition, the locally free sheaf $\mathcal{E}_G$ is a finite locally free $\mathcal{O}/z^n \otimes_{\mathbb{F}_q} \mathcal{O}_S$-module. It follows that $\mathcal{D}(G)$ is also a finite locally free $\mathcal{O}/z^n \otimes_{\mathbb{F}_q} \mathcal{O}_S$-module, which is killed by z^n and therefore, it is a z-sheaf of degree n. It is now clear that the functor $G \mapsto \mathcal{D}(G)$ induces an equivalence of categories, preserving degrees, from the category of finite z-modules to the category of z-sheaves. ◊

Proposition 3.18. *The functor $G \mapsto (\mathcal{D}(G), v_G, \varphi_G)$ is an equivalence of categories from the category of (strict) finite v-modules over S to the category of (strict) Frob-sheaves over S.*

Proof. This follows from Proposition 3.12. □

Proposition 3.19. *Let G_1, G_2 and G_3 be finite φ-modules over S. For each i, let $\mathcal{E}_i$ and respectively $\mathcal{D}_i$ be the finite $\mathbb{F}_q$-shtuka and Ver-sheaf attached to G_i. Consider the sequences*

$$0 \to G_1 \to G_2 \to G_3 \to 0 \tag{3.20}$$

$$0 \to \mathcal{E}_3 \to \mathcal{E}_2 \to \mathcal{E}_1 \to 0 \tag{3.21}$$

$$0 \to \mathcal{D}_1 \to \mathcal{D}_2 \to \mathcal{D}_3 \to 0 \tag{3.22}$$

that are obtained by functoriality or duality one from the other, and the first one is regarded as a sequence of fppf sheaves and the other two are regarded as sequences of $\mathcal{O}_S$-modules. Then, if one sequence is exact the other two are exact as well.

Proof. Note that the sequence (3.21) is exact if and only if the sequence (3.22) is exact. Indeed, we can assume that S is the spectrum of a local ring and so $\mathcal{E}_1$ and $\mathcal{D}_3$ are finite free $\mathcal{O}_S$-modules. Then, the exactness of these sequences is equivalent to them being split. Now since $\mathcal{D}_i$ is the $\mathcal{O}_S$-dual, the claim follows.

Now, assume that the sequence (3.20) is exact. We want to show that the sequence (3.21) is exact. Let T be an S-scheme. The sequence

$$0 \to G_{1,T} \to G_{2,T} \to G_{3,T} \to 0$$

obtained by base change to T is also exact. The functor $\mathrm{Hom}_{\mathbb{F}_q,T}(\_, \mathbb{G}_a)$ on the category of fppf sheaves over T is left exact, and so is $\underline{\mathrm{Hom}}_{\mathbb{F}_q,T}(\_, \mathbb{G}_a)$. This shows the exactness of the sequence

$$0 \to \mathcal{E}_{3,T} \to \mathcal{E}_{2,T} \to \mathcal{E}_{1,T} \tag{3.23}$$

for every S-scheme T.

It remains to show that $\mathcal{E}_2 \to \mathcal{E}_1$ is an epimorphism of Zariski sheaves. We can therefore assume that $S = \mathbf{Spec}(R)$ is the spectrum of a local ring $(R, \mathfrak{m}, \kappa)$

and that $\mathcal{E}_i$ $(i = 1, 2, 3)$ are finite free R-modules. By Nakayama's lemma and (3.23), we can even assume that $R = \kappa$ is a field and so $\mathcal{E}_i$ $(i = 1, 2, 3)$ are finite dimensional κ-vector spaces. The exactness of $0 \to G_1 \to G_2 \to G_3 \to 0$ implies that $|G_2| = |G_1||G_3|$ and so $\dim_\kappa \mathcal{E}_2 = \dim_\kappa \mathcal{E}_1 + \dim_\kappa \mathcal{E}_3$ (see Remark 1.6). This implies that $\mathcal{E}_2 \to \mathcal{E}_1$ is surjective, and therefore the sequence (3.21) is exact as desired.

Conversely, assume that (3.21) is exact. We want to show that (3.20) is exact, and again, we can assume that $S = \mathbf{Spec}(R)$ is the spectrum of a local ring. Thus, the sequence (3.21) is split and so we can find a basis $\{x_1, \ldots, x_n\}$ of $\mathcal{E}_3$ and extend it to a basis $\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ of $\mathcal{E}_2$. Let us first show that $G_2 \to G_3$ is an epimorphism of fppf sheaves, by showing that the morphism on the affine algebras is faithfully flat. By construction of G_i (see the proof of Proposition 3.9), we have

$$\begin{aligned} \mathcal{O}_{G_3} &\cong \frac{R[X_1, \ldots, X_n]}{\big(X_i^q - \varphi_3(X_i)\big)} \\ \mathcal{O}_{G_2} &\cong \frac{R[X_1, \ldots, X_n, Y_1, \ldots, Y_m]}{\big(X_i^q - \varphi_3(X_i), Y_j^q - \varphi_2(Y_j)\big)} \end{aligned} \tag{3.24}$$

where φ_i is the Frobenius of $\mathcal{E}_i$. The R-module $\mathcal{O}_{G_2}$ is free of rank q^{n+m} and the set $\{X_i^{r_i} Y_j^{s_j} \,|\, 0 \leq r_i, s_j \leq q-1\}$ generates it as an R-module, and therefore is a basis. It implies that the set $\{Y_j^{s_j} \,|\, 0 \leq s_j \leq q-1\}$ is a basis of $\mathcal{O}_{G_2}$ over $\mathcal{O}_{G_3}$ (note that the set $\{X_i^{r_i} \,|\, 0 \leq r_i \leq q-1\}$ is a basis of $\mathcal{O}_{G_3}$ over R) and therefore $\mathcal{O}_{G_3} \to \mathcal{O}_{G_2}$ is faithfully flat. This shows that $G_2 \to G_3$ is an epimorphism of fppf sheaves and a flat morphism of schemes. Therefore, its scheme-theoretic kernel, denoted by G' is a finite φ-module (being the pullback along the identity section, it is flat and so a finite φ-module). So, the sequence

$$0 \to G' \to G_2 \to G_3 \to 0 \tag{3.25}$$

is an exact sequence of fppf sheaves.

By functorialty, the composition $G_1 \to G_2 \to G_3$ is trivial, since $\mathcal{E}_3 \to \mathcal{E}_2 \to \mathcal{E}_1$ is trivial. This implies that $G_1 \to G_2$ factors through $G' \to G_2$:

$$\begin{array}{ccc} G_1 & \xrightarrow{\qquad\qquad} & G_2 \\ & {\scriptstyle u}\searrow \qquad \nearrow & \\ & G' & \end{array} \tag{3.26}$$

By the first part of the proof, the sequence (3.25) yields an exact sequence

$$0 \to \mathcal{E}_3 \to \mathcal{E}_2 \to \mathcal{E}_{G'} \to 0$$

of R-modules. But then $\mathcal{E}_u : \mathcal{E}_{G'} \to \mathcal{E}_1$ is an isomorphism, as (3.21) is also exact. This shows that $u : G_1 \to G'$ is an isomorphism, and so, the sequence (3.20) is exact as desired. □

Proposition 3.27. *Let X be an $\mathbb{F}_q$-scheme. Quotient of monomorphisms and kernels of epimorphisms of finite φ-modules exists. More precisely, let G, H be finite φ-modules over X, then*

(a) if $\iota : G \hookrightarrow H$ is a monomorphism of fppf sheaves, then there is a finite φ-module C over X and a short exact sequence

$$0 \to G \xrightarrow{\iota} H \to C \to 0 \tag{3.28}$$

of finite φ-modules.

(b) if $\pi : G \twoheadrightarrow H$ is an epimorphism of fppf sheaves, then there is a finite φ-module K over X and a short exact sequence

$$0 \to K \to G \xrightarrow{\pi} H \to 0 \tag{3.29}$$

of finite φ-modules.

Proof. (a) By Proposition 3.19, $\mathcal{E}_H \to \mathcal{E}_G$ is an epimorphism of $\mathcal{O}_S$-modules. Let $\mathcal{E}$ be its kernel as a Zariski sheaf. The short exact sequence

$$0 \to \mathcal{E} \to \mathcal{E}_H \to \mathcal{E}_G \to 0$$

splits over sufficiently small open subschemes of X, as $\mathcal{E}_G$ is locally free and so $\mathcal{E}_H \cong \mathcal{E} \oplus \mathcal{E}_G$ (note that in order to show that $\mathcal{E}$ is locally free, we can restrict to a covering of X by Zariski open subschemes). Since $\mathcal{E}_H$ is finite locally free and $\mathcal{E}$ is a direct summand, it is finite locally free as well. Here we are using the fact that over a ring finite projective modules are the same as finite locally free modules, and that direct summands of projective modules are projective. Now, since the projection $\mathcal{E}_H \twoheadrightarrow \mathcal{E}_G$ is a morphism of finite $\mathbb{F}_q$-shtukas, Frobenius of $\mathcal{E}_H$ induces a Frobenius on $\mathcal{E}$, being the kernel of $\mathcal{E}_H \twoheadrightarrow \mathcal{E}_G$. This shows that indeed $\mathcal{E}$ is a finite $\mathbb{F}_q$-shtuka over X. Now, let C be the finite φ-module corresponding to $\mathcal{E}$ (see Proposition 3.9). We therefore have a complex

$$0 \to G \to H \to C \to 0 \tag{3.30}$$

By Proposition 3.19 this is a short exact sequence of fppf sheaves.

(b) Let K be the scheme-theoretic kernel of π (which is the kernel as an fppf sheaf). It suffices to show that K is flat over S, because then as G is finite over S and embeds in some vector group scheme, K also has such an embedding and it will be finite φ-module giving rise to the desired short exact sequence (3.29). Since G is finite flat over S, the criterion of flatness along the fibers (see [7], Corollaire 11.3.11) implies that $\pi : G \to H$ is a flat morphism and so, K being its base change along the identity section, it is flat over S.

□

Definition 3.31. The *Lubin-Tate group*[2] $\mathcal{LT}$ over S is a strict $\mathcal{O}$-module scheme over S with underlying $\mathbb{F}_q$-module scheme $\mathbb{G}_a$, on which z acts by $x \mapsto \zeta x + x^q$. Let us write the polynomial $(X^q + \zeta X)^{\circ n} \in \mathbb{F}_q[\zeta][X]$ (here the notation "$\circ\, n$" means composing the polynomial n times with itself) as

$$X^{q^n} + y_{n-1}X^{q^{n-1}} + \cdots + y_1 X^q + y_0 X$$

with $y_i \in \mathbb{Z}[\zeta]$ (e.g., $y_0 = \zeta^n, y_{n-1} = \zeta + \zeta^q + \cdots + \zeta^{q^{n-1}}$).

Let G be the finite subgroup $\mathcal{LT}[z^n]$ of $\mathcal{LT}$. Then $\mathcal{D}(G)$ is a free $\mathcal{O}_S$-module of rank n with basis $\xi_i, i = 0, \ldots, n-1$ (the dual basis of X^{q^i} under the duality between $\mathcal{D}(G)$ and $\mathcal{E}_G$). The finite φ-module G has a unique structure of a finite v-module, given by the following structure of a Frob-sheaf on $\mathcal{D}(G)$: $\forall\, i = 0, \ldots, n-1$, and with the convention that $\xi_{-1} = 0 = \xi_n$, we have

- $z \cdot \xi_i = \xi_{i-1} + \zeta^{q^i}\xi_i - y_i\xi_{n-1}$
- $v(\xi_i) = \xi_{i-1} \otimes 1 - y_i\xi_{n-1} \otimes 1$
- $\varphi(\xi_i \otimes 1) = \xi_i + (\zeta^{q^{i+1}} - \zeta)\xi_{i+1}$.

▲

Remark 3.32. 1) It is straightforward to show that the inclusion $\mathcal{LT}[z^n] \hookrightarrow \mathcal{LT}[z^{n+1}]$ is a homomorphism of finite v-modules.

2) Let G be a finite v-module over S. Then, G is killed by a power of z and therefore the inclusion $\mathrm{Hom}(G, \mathcal{LT}[z^n]) \hookrightarrow \mathrm{Hom}(G, \mathcal{LT})$ is a bijection for n large enough. We denote by $\mathrm{Hom}_V(G, \mathcal{LT})$ the subgroup $\mathrm{Hom}_V(G, \mathcal{LT}[z^n])$ of $\mathrm{Hom}(G, \mathcal{LT})$ (which is independent of the choice of $n >> 0$). ♢

Proposition 3.33 (Theorem (4.3), [14])**.** *The functor*

$$G \mapsto \underline{\mathrm{Hom}}_V(G, \mathcal{LT})$$

[2] This is called Carlitz module in [14].

induces a self-duality on the category of finite v-modules over S. In other words, for any finite v-module G, the functor

$$\underline{\mathrm{Hom}}_V(G, \mathcal{LT}) : \mathfrak{Sch}_S \to \mathfrak{Mod}_{\mathcal{O}}$$

that sends an S-scheme T to the $\mathcal{O}$-module $\mathrm{Hom}_V(G_T, \mathcal{LT}_T)$ is representable by a finite v-module over S and the natural pairing

$$G \times_S \underline{\mathrm{Hom}}_V(G, \mathcal{LT}) \to \mathcal{LT}$$

is perfect.

Remark 3.34. In [14], this theorem is stated and proved only for the case of strict finite v-modules, but the proof works for general finite v-modules, and therefore we don't repeat it here. ♢

Definition 3.35. Let G be a finite v-module over S. We denote the dual v-module $\underline{\mathrm{Hom}}_V(G, \mathcal{LT})$ by G^* and call it *the Carlitz dual* of G. ▲

Remark 3.36. 1) One can show (cf. [14], §4) that there is a canonical isomorphism

$$\mathcal{D}(G^*) \cong \underline{\mathrm{Hom}}_{\mathcal{O}_S}(\mathcal{D}(G), \mathcal{O}_S).$$

Under this isomorphism, Verschiebung and Frobenius are switched. This is similar to the Cartier duality for finite flat group schemes and one can think of the Lubin-Tate group $\mathcal{LT}$ as the analogue of the multiplicative group. A similar duality has been worked out for general finite flat strict $\mathcal{O}$-group schemes by Faltings (cf. [5], §5)

2) Assume that ζ is zero in $\mathcal{O}_S$. Then $F_G : G \to G^{(q)}$ is a map of finite v-modules over S and therefore, the Verschiebung $V_G : \mathbb{V}_G^{(q)} \to \mathbb{V}_G$ restricts to a morphism $G^{(q)} \to G$ of finite φ-modules that we still denote by V_G (note that φ_G and v_G commute up to Frobenius twist). This map is the Carlitz dual of F_{G^*} and we have $F_G \circ V_G = z$ and $V_G \circ F_G = z$.

♢

3.2 Drinfeld displays

Definition 3.37. (i) A *Drinfeld display* of *rank h* over S is a pair $\mathcal{P} = (P, V)$, where

- P is a sheaf of $\mathcal{O}_S[\![z]\!]$-modules over S that is, Zariski locally on S, a free $\mathcal{O}_S[\![z]\!]$-module of rank h and
- $V : P \to \sigma^* P$ is a an $\mathcal{O}_S[\![z]\!]$-linear map, such that $V[\frac{1}{z-\zeta}]$ is an isomorphism.

Morphisms of Drinfeld displays are the obvious ones.

(ii) $\mathcal{P}$ is said to be *nilpotent*, if V is nilpotent modulo z, meaning that there is an $n \geq 1$ such that the composition $V^n := \sigma^{n*}V \circ \cdots \circ \sigma^* V \circ V$ is zero modulo z (i.e., $V^n(P) \subseteq z(\sigma^{n*}P)$).

(iii) $\mathcal{P}$ is said to be *étale* if V is an isomorphism.

(iv) The *Lie algebra* or *tangent space* of $\mathcal{P}$ is defined to be the cokernel of V. ▲

Remark 3.38. Let $\mathcal{P} = (P, V)$ be a Drinfeld display over S.

1) Let $a : T \to S$ be an S-scheme. Then $\mathcal{P}_T := (P \otimes_{\mathcal{O}_S[\![z]\!]} \mathcal{O}_T[\![z]\!], V \otimes \mathrm{Id})$ is a Drinfeld display over T, called the *base change* of $\mathcal{P}$ to T, and it has the same rank and dimension as $\mathcal{P}$. We sometimes denote $\mathcal{P}_T$ by $a^*\mathcal{P}$ when we want to emphasize the structure morphism a.

2) The condition that $V[\frac{1}{z-\zeta}]$ be an isomorphism, is equivalent to V being injective and that there be an $N \geq 1$ such that $(z-\zeta)^N$ annihilates the cokernel of P. In this case, there exists a unique $\mathcal{O}[\![z]\!]$-linear map $F : \sigma^*P \to P$ such that

$$V \circ F = (z-\zeta)^N.$$

We have

$$V \circ F \circ V = (z-\zeta)^N V = V(z-\zeta)^N$$

and since V is injective, we conclude that $F \circ V = (z-\zeta)^N$. The number N is called the *degree of nilpotence* of $\mathcal{P}$. If $N = 1$, we call $\mathcal{P}$ *strict*. ♢

Proposition 3.39. *The tangent space of a Drinfeld display over S is a locally free $\mathcal{O}_S$-module of finite rank.*

Proof. Let us denote the cokernel of V by L. We can assume that $S = \mathbf{Spec}(R)$ is affine, with $(R, \mathfrak{m})$ a local ring. Let κ be the residue field of R. Then, $R[\![z]\!]$ is a local ring (with maximal ideal generated by $\mathfrak{m}$ and z) and so P is a free $R[\![z]\!]$-module.

To simplify the notations, let us write Q for σ^*P. Consider the following short exact sequence of $R[\![z]\!]$-modules (note that by the above remark, V is injective):

$$0 \to P \xrightarrow{V} Q \to L \to 0 \tag{3.40}$$

Here P and Q are free $R[\![z]\!]$-modules of the same rank, say h. As it will be clear from the following arguments, the fact that L is a finite free R-module follows from the fact that it is the cokernel of an injective homomorphism of free $R[\![z]\!]$-modules of the same rank and it is killed by a power of $z-\zeta$ (as was shown in the previous remark).

Since by assumption ζ is nilpotent in R, the morphism

$$\begin{aligned} R[\![z]\!] &\to R[\![z]\!] \\ z &\mapsto z-\zeta \end{aligned}$$

is an isomorphism of topological rings. Since L is killed by a power of $z-\zeta$, by twisting all modules (i.e., P, Q and L) with the above isomorphism, we can assume that L is killed by a power of z, say z^N.

Modulo z^N, we have the following exact sequence of $R[\![z]\!]/z^N$-modules (note that $L/z^N = L$)

$$P/z^N \xrightarrow{V} Q/z^N \to L \to 0 \tag{3.41}$$

As P/z^N and Q/z^N are finite free $R[\![z]\!]/z^N$-modules, this implies that L is finitely presented. Now assume that $\mathrm{Tor}_1^R(\kappa, L) = 0$. Pick elements $x_1, \ldots, x_d \in L$ whose images in $L \otimes_R \kappa$ form a basis over κ. Then, by Nakayama's lemma, we have an epimorphism

$$R^d \twoheadrightarrow L$$

and since L is finitely presented, the kernel of this epimorphism, K, is finitely generated. Tensoring the short exact sequence

$$o \to K \to R^d \to L \to 0$$

with κ yields

$$0 \to \mathrm{Tor}_1^R(\kappa, L) \to K \otimes_R \kappa \to \kappa^d \to L \otimes_R \kappa \to 0$$

and since by assumption $\mathrm{Tor}_1^R(\kappa, L) = 0$, $\kappa^d \to L \otimes_R \kappa$ is an isomorphism and K is finitely generated, by Nakayama's lemma, we have that $K = 0$ and so L is free. Therefore, it is enough to show that $\mathrm{Tor}_1^R(\kappa, L) = 0$.

Tensoring short exact sequence (3.40) with $\kappa[\![z]\!]$ yields

$$0 \to \mathrm{Tor}_1^{R[\![z]\!]}(\kappa[\![z]\!], L) \to P \otimes_{R[\![z]\!]} \kappa[\![z]\!] \xrightarrow{V \otimes 1} Q \otimes_{R[\![z]\!]} \kappa[\![z]\!] \to L \otimes_{R[\![z]\!]} \kappa[\![z]\!] \to 0$$

Since $z^N L = 0$, the cokernel of $P \otimes_{R[\![z]\!]} \kappa[\![z]\!] \xrightarrow{V \otimes 1} Q \otimes_{R[\![z]\!]} \kappa[\![z]\!]$ is z^N-torsion. So, by elementary divisors theorem, the image of $V \otimes 1$ has rank equal to the rank of $Q \otimes_{R[\![z]\!]} \kappa[\![z]\!]$, and since $P \otimes_{R[\![z]\!]} \kappa[\![z]\!]$ also has the same rank, this implies that $V \otimes 1$ is injective. Thus, $\mathrm{Tor}_1^{R[\![z]\!]}(\kappa[\![z]\!], L) = 0$. By the base change theorem for Tor (Theorem 5.6.6, p.144, [15]) and the homology exact sequence of low-degree terms (Corollary 5.8.4, p.151, [15]) we have an epimorphism

$$\mathrm{Tor}_1^{R[\![z]\!]}(\kappa[\![z]\!], L) \twoheadrightarrow \mathrm{Tor}_1^{R[\![z]\!]/z^N}(\kappa[\![z]\!]/z^N, L)$$

and so $\mathrm{Tor}_1^{R[\![z]\!]/z^N}(\kappa[\![z]\!]/z^N, L) = 0$ as well. Since $R[\![z]\!]/z^N$ is a flat R-module, the flat base change for Tor (Proposition 3.2.9, p.72, [15]) implies that

$$\mathrm{Tor}_1^{R[\![z]\!]/z^N}(\kappa[\![z]\!]/z^N, L) \cong \mathrm{Tor}_1^R(\kappa, L)$$

and so $\mathrm{Tor}_1^R(\kappa, L) = 0$ as desired. □

Remark 3.42. A similar statement about local shtukas with a similar proof is given in [8]. ♢

Definition 3.43. The *dimension* of a Drinfeld display over S is the rank of its tangent space as a finite locally free $\mathcal{O}_S$-module. ▲

Remark 3.44. Let $\mathcal{P} = (P, V)$ be a Drinfeld display over S of rank h. Let us denote the quotient P/z^nP by P_n. This is a finite locally free $\mathcal{O}_S[\![z]\!]/z^n$-module of rank h. Since $\mathcal{O}_S[\![z]\!]/z^n$ is a finite free $\mathcal{O}_S$-module of rank n, it follows that P_n is a finite locally free $\mathcal{O}_S$-module of rank nh. The $\mathcal{O}_S[\![z]\!]$-linear morphism $V : P \to \sigma^*P$ induces an $\mathcal{O}_S$-linear morphism $V : P_n \to \sigma^*P_n$ that we still denote by V. Therefore, the pair (P_n, V) is a Ver-sheaf killed by z^n and by what we said above, it is in fact a z-sheaf of degree n. By previous remark we also have a map $F : \sigma^*P \to P$ such that $F \circ V = V \circ F = (z - \zeta)^N$ for some N. So, F induces a map $\sigma^*P_n \to P_n$ such that $F \circ V = V \circ F = (z - \zeta)^N$. Thus, (P_n, V, F) is a Frob-sheaf, and it is strict if $\mathcal{P}$ is so. ♢

Definition 3.45. Let G be a z-divisible local Anderson module over S. Define the *Drinfeld display* of G to be the $\mathcal{O}_S[\![z]\!]$-module $\mathcal{D}(G) := \varprojlim_n \mathcal{D}(G_n)$, where the transition morphisms are induced by the projections $z : G_{n+1} \twoheadrightarrow G_n$. Each G_n is an $\mathcal{O}$-module scheme, and so, by functoriality, $\mathcal{D}(G_n)$ is an $\mathcal{O}_S[\![z]\!]$-module. The Verschiebungen $v : \mathcal{D}(G_n) \to \sigma^*\mathcal{D}(G_n)$ induce an $\mathcal{O}_S[\![z]\!]$-linear morphism $V_G : \mathcal{D}(G) \to \sigma^*\mathcal{D}(G)$. ▲

Lemma 3.46. *Let R be a local ring and*

$$0 \to K \to M \to N \xrightarrow{\rho} C \to 0$$

an exact sequence where M, N, C are finite free R-modules. Then, we have an isomorphism $M \cong K \oplus \mathrm{Ker}(\rho)$, and so, K is a finite free R-module as well. Moreover, we have

$$\mathrm{rank}(K) = \mathrm{rank}(M) - \mathrm{rank}(N) + \mathrm{rank}(C)$$

Proof. Let us denote $\mathrm{Ker}(\rho)$ by I. Since C is projective (free), the short exact sequence

$$0 \to I \to N \to C \to 0$$

splits and we have an isomorphism

$$N \cong C \oplus I$$

and so I is finite projective, and since R is local, it is free and its rank is $\mathrm{rank}(N) - \mathrm{rank}(C)$. Now, since I is projective, the short exact sequence

$$0 \to K \to M \to I \to 0$$

splits and we have

$$M \cong K \oplus I$$

as desired. This implies that K is finite projective and therefore free. Its rank is equal to

$$\mathrm{rank}(M) - \mathrm{rank}(I) = \mathrm{rank}(M) - \mathrm{rank}(N) + \mathrm{rank}(C)$$

□

Proposition 3.47. *Let G be a z-divisible local Anderson module over S, of height h and dimension d. Then $(\mathcal{D}(G), V_G)$ is a Drinfeld display of rank h and dimension d. If G is infinitesimal (respectively étale), then $\mathcal{D}(G)$ is nilpotent (respectively étale).*

Proof. We have to show that $\mathcal{D}(G)$ is a locally free $\mathcal{O}_S[\![z]\!]$-module of rank h, that V is injective and its cokernel of V is killed by a power of $z - \zeta$ and that the rank of the cokernel of V as a locally free $\mathcal{O}_S$-module is equal to d.

Let us, as usual, denote by G_n the kernel of $z^n : G \to G$. Let us also denote by $\mathcal{D}_n$ the Ver-sheaf $\mathcal{D}(G_n)$. Then, by Remark 1.12, for all $i, n \geq 0$, we have an exact sequence

$$0 \to G_i \xrightarrow{z^n} G_{n+i} \to G_n \to 0$$

By Proposition 3.19, the induced sequence

$$0 \to \mathcal{D}_i \xrightarrow{z^n} \mathcal{D}_{n+i} \to \mathcal{D}_n \to 0$$

is exact. Since for all $m \geq 0$, the transition morphisms $\mathcal{D}_{m+i+1} \to \mathcal{D}_{m+i}$ induced by the epimorphisms $G_{m+i+1} \twoheadrightarrow G_{m+i}$ are epimorphisms, by Proposition 3.19, the projective system

$$\mathcal{D}_n \leftarrow \mathcal{D}_{n+1} \leftarrow \cdots$$

satisfies the Mittag-Leffler condition and therefore, the sequence

$$0 \to \varprojlim_i \mathcal{D}_i \xrightarrow{z^n} \varprojlim_i \mathcal{D}_{n+i} \to \mathcal{D}_n \to 0$$

is exact. Note that we have $\mathcal{D}(G) = \varprojlim_i \mathcal{D}_i \cong \varprojlim_i \mathcal{D}_{n+i}$ and that the composition

$$\varprojlim_i \mathcal{D}_i \xrightarrow{z^n} \varprojlim_i \mathcal{D}_{n+i} \cong \varprojlim_i \mathcal{D}_i$$

is nothing but the multiplication with z^n. So, we have a short exact sequence of $\mathcal{O}_S[\![z]\!]$-modules

$$0 \to \mathcal{D}(G) \xrightarrow{z^n} \mathcal{D}(G) \to \mathcal{D}_n \to 0 \tag{3.48}$$

We are now going to first show that $\mathcal{D}(G)$ is, Zariski locally, a free $\mathcal{O}_S[\![z]\!]$-module of rank h. Then, we will show that V is injective, and that its cokernel is canonically isomorphic, as an $\mathcal{O}_S[\![z]\!]$-module, to the tangent space of G. Since by definition the tangent space of G is killed by a power of $z - \zeta$ and has rank d as a locally free $\mathcal{O}_S$-module, the proof is achieved. We can assume that $S = \mathbf{Spec}(R)$, where $(R, \mathfrak{m}, \kappa)$ is a local ring. Then, $R[\![z]\!]$ is a local ring with maximal ideal $(\mathfrak{m}, z)$ and residue field κ. By (3.48), we have

$$\mathcal{D}(G) \otimes_{R[\![z]\!]} \kappa \cong \mathcal{D}_1 \otimes_R \kappa$$

Since G_1 is a finite φ-module of order q^h, its Ver-sheaf $\mathcal{D}_1$ is a free R-module of rank h and so $\mathcal{D}_1 \otimes_R \kappa$ is a κ-vector space of dimension h. Let $x_1, \dots, x_h \in \mathcal{D}(G)$ be elements that map to a basis of $\mathcal{D}(G) \otimes_{R[\![z]\!]} \kappa$. For all $n \geq 0$, consider the images of x_i in $\mathcal{D}(G)/z^n \cong \mathcal{D}_n$. As $\mathcal{D}_n$ is a finitely generated R-module, it is finite as an $R[\![z]\!]/z^n$-module. So, by Nakayama's lemma, x_i generate $\mathcal{D}_n$ and we have an epimorphism

$$\begin{aligned} (R[\![z]\!]/z^n)^{\oplus h} &\to \mathcal{D}_n \to 0 \\ e_i &\mapsto x_i \end{aligned}$$

Since both $\mathcal{D}_n$ and $(R[\![z]\!]/z^n)^{\oplus h}$ are free R-modules of rank nh, it follows that this epimorphism is in fact injective and so, is an isomorphism of $R[\![z]\!]$-modules. Therefore, its inverse limit yields an isomorphism

$$\begin{aligned} R[\![z]\!]^{\oplus h} &\xrightarrow{\cong} \mathcal{D}(G) \\ e_i &\mapsto x_i \end{aligned}$$

For all $n \geq 1$, let us denote by K_n and respectively ω_n the kernel and respectively cokernel of $\varphi_n : \sigma^* \mathcal{E}_{G_n} \to \mathcal{E}_{G_n}$. By Remark 3.8 1), this the dual of the tangent space of G_n. We also know that there is an $n_0 \gg 0$ such that for all $n \geq n_0$, the morphism $\omega_{n+1} \to \omega_n$ induced by the inclusion $G_n \hookrightarrow G_{n+1}$ is an isomorphism of finite free R-modules of rank d (see Remark 1.14). So, for $n \geq n_0$ and $i \geq n$, the morphism $\omega_n \to \omega_{n+i}$ induced by the projection $G_{n+i} \twoheadrightarrow G_n$ is trivial (because the composition $\omega_n \to \omega_{n+i} \to \omega_n$ is the multiplication by z^i, which is trivial on ω_n). Therefore, for large n and i, the morphism $\omega_{n+i}^\vee \to \omega_n^\vee$ is trivial. Now, for every $n \geq n_0$ consider the exact sequence

$$0 \to K_n \to \sigma^* \mathcal{E}_{G_n} \xrightarrow{\varphi_n} \mathcal{E}_{G_n} \to \omega_n \to 0 \tag{3.49}$$

By Lemma 3.46, K_n is a finite free R-module of rank d and the dual sequence

$$0 \to \omega_n^\vee \to \mathcal{D}_n \xrightarrow{v_n} \sigma^* \mathcal{D}_n \to K_n^\vee \to 0 \tag{3.50}$$

is exact as well. It also follows from the lemma that the inverse limit of this sequence remains exact:

$$0 \to \varprojlim_n \omega_n^\vee \to \mathcal{D}(G) \xrightarrow{V} \sigma^* \mathcal{D}(G) \to \varprojlim_n K_n^\vee \to 0 \tag{3.51}$$

The inverse limit $\varprojlim_n \omega_n^\vee$ is zero, as for large n, i, the transition morphisms $\omega_{n+i}^\vee \to \omega_n^\vee$ are trivial. This shows that V is injective. Now, for $n \gg 0$, the cokernel $K_n^\vee$ of $v_n : \mathcal{D}_n \to \sigma^* \mathcal{D}_n$ is finite free of rank d and we have surjections $K_{n+1}^\vee \twoheadrightarrow K_n^\vee$. This implies that in fact these surjections are isomorphisms and that the projection

$$\varprojlim_n K_n^\vee \to K_n^\vee$$

is an isomorphism for $n \gg 0$. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{D}(G) & \xrightarrow{z^n} & \mathcal{D}(G) & \longrightarrow & \mathcal{D}_n & \longrightarrow & 0 \\ & & \downarrow V & & \downarrow V & & \downarrow v_n & & \\ 0 & \longrightarrow & \mathcal{D}(G) & \xrightarrow{z^n} & \mathcal{D}(G) & \longrightarrow & \mathcal{D}_n & \longrightarrow & 0 \end{array} \tag{3.52}$$

Since V is injective, the Snake Lemma yields the following exact sequence of $R[\![z]\!]$-modules (for $n \gg 0$)

$$0 \to \omega_n^\vee \to \mathrm{Coker}(V) \xrightarrow{z^n} \mathrm{Coker}(V) \xrightarrow{\cong} K_n^\vee \to 0 \tag{3.53}$$

Therefore, the $\mathrm{Coker}(V)$ is canonically isomorphic to $\omega_n^\vee$, which is by definition the tangent space of G.

The last statement of the proposition follows from the fact that if G is infinitesimal, then $v : \mathcal{D}_1 \to \sigma^*\mathcal{D}_1$ is nilpotent. But $\mathcal{D}_1$ is $\mathcal{D}(G)/z$, and so, $V : \mathcal{D}(G) \to \sigma^*\mathcal{D}$ is nilpotent modulo z and so $\mathcal{D}(G)$ is nilpotent. Similarly, if G is étale, then all $v_n : \mathcal{D}_n \to \sigma^*\mathcal{D}_n$ are isomorphisms and so $V : \mathcal{D}(G) \to \sigma^*\mathcal{D}(G)$ is an isomorphism and so $\mathcal{D}(G)$ is étale. □

Remark 3.54.
1) We showed in the proof of the above proposition that in fact the tangent space of $\mathcal{D}(G)$ is canonically isomorphic to the tangent space of G.
2) We also proved that if G is of height h, then for all $n \geq 1$, the Ver-sheaf $\mathcal{D}(G_n)$, which has a canonical structure of a $\mathcal{O}_S[\![z]\!]/z^n$-module (note that $\mathcal{O} \otimes_{\mathbb{F}_q} \mathcal{O}_S \cong \mathcal{O}_S[\![z]\!]/z^n$) is the cokernel of multiplication by z^n on $\mathcal{D}(G)$ and is a locally free $\mathcal{O}_S[\![z]\!]/z^n$-module of rank h. Therefore, it is a z-sheaf of degree n, and G_n is a finite z-module of degree n. ♢

Construction 3.55. Let $\mathcal{P} = (P, V)$ be a Drinfeld display over S. We are going to attach to $\mathcal{P}$ an fppf sheaf of $\mathcal{O}$-modules $DA(\mathcal{P})$. Define a functor $DA(\mathcal{P})_n : \mathfrak{Sch}_S \to \mathfrak{Mod}_{\mathcal{O}}$ by sending an S-scheme T to the $\mathcal{O}$-module

$$(P/z^n \otimes_{\mathcal{O}_S} \mathcal{O}_T)^{V=1} :=$$

$$\big\{x \in P/z^n \otimes_{\mathcal{O}_S} \mathcal{O}_T \mid (V \otimes \mathrm{Id})(x) = x \otimes 1 \in \sigma_T^*(P/z^n \otimes_{\mathcal{O}_S} \mathcal{O}_T)\big\}.$$

This is an $\mathcal{O}$-module, where $\mathbb{F}_q$ acts via the $\mathcal{O}_S$-module structure of P (and the inclusion $\mathbb{F}_q \hookrightarrow \mathcal{O}_S$) and the action of z is through its action on P. Now set $DA(\mathcal{P}) := \varinjlim_n DA(\mathcal{P})_n$. In fact, one can see that $DA(\mathcal{P})$ is the limit $\varinjlim_n \mathcal{G}(P_n)$, where P_n is the Ver-sheaf P/z^n (see Remark 3.44) and $\mathcal{G}(P_n)$ is the finite φ-module attached to it (see Proposition 3.12). ▼

Proposition 3.56. *Let $\mathcal{P} = (P, V)$ be a Drinfeld display over S, of rank h and dimension d. Then $DA(\mathcal{P})$ is a z-divisible local Anderson module of height h and dimension d. Moreover, if $\mathcal{P}$ is nilpotent (respectively étale) then $DA(\mathcal{P})$ is infinitesimal (respectively étale).*

Proof. Let us denote by P_n the Ver-sheaf $(P/z^n, V)$ and G_n the corresponding finite φ-module. For every $n \geq 1$, multiplication by z^n yields an exact sequence

$$0 \to P_n \xrightarrow{z\cdot} P_{n+1} \xrightarrow{z^n} P_{n+1}$$

The cokernel of $P_n \xrightarrow{z\cdot} P_{n+1}$ is P_1, which is a Ver-sheaf, and so, by Proposition 3.19, the corresponding sequence

$$0 \to G_n \to G_{n+1} \xrightarrow{z^n} G_{n+1}$$

is exact as well. The rank of P_n, as a free $\mathcal{O}_S$-module, is nh and so, the order of G_n is q^{nh}. By Remark 1.13, the limit $G := \varinjlim_n G_n$ will be a z-divisible local Anderson module of height h, if we show that the action of $z-\zeta$ on the tangent space $\mathcal{L}ie\, G := \varinjlim_n \mathcal{L}ie\, G_n$ is locally nilpotent. Recall that $\mathcal{L}ie\, G_n$ is canonically isomorphic to the $\mathcal{O}_S$-dual of the cokernel of $\varphi : \sigma^*\mathcal{E}_{G_n} \to \mathcal{E}_{G_n}$, and so is canonically isomorphic to the kernel of $V : P_n \to \sigma^* P_n$. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P & \xrightarrow{z^n} & P & \longrightarrow & P_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle V} & & \downarrow{\scriptstyle V} & & \downarrow{\scriptstyle V} & & \\ 0 & \longrightarrow & P & \xrightarrow{z^n} & P & \longrightarrow & P_n & \longrightarrow & 0 \end{array}$$

Since $V : P \to P$ is injective, by Snake Lemma we have an injection

$$\mathcal{L}ie\, G_n \hookrightarrow \mathrm{Coker}(V)$$

of $\mathcal{O}_S[\![z]\!]$-modules. Since a power of $z-\zeta$ kills $\mathrm{Coker}(V)$, it kills $\mathcal{L}ie\, G_n$ and the limit $\mathcal{L}ie\, G$ as well. Hence G is a z-divisible local Anderson module of height h. Now, we have to show that its dimension is d.

Since P is a finite locally free $\mathcal{O}_S[\![z]\!]$-module, and $\mathcal{O}_S[\![z]\!]$ is z-adically complete, the projections $P \twoheadrightarrow P_n$ induce a canonical isomorphism

$$P \cong \varprojlim_n P_n$$

But we have canonical and compatible (with $G_{n+1} \twoheadrightarrow G_n$ and $P_{n+1} \twoheadrightarrow P_n$) isomorphisms $P_n \cong \mathcal{D}(G_n)$, which imply that $\mathcal{P}$ is a the Drinfeld display $\mathcal{D}(G)$ attached to G. It now follows from Proposition 3.47 and Remark 3.54 1) that the tangent spaces of $\mathcal{P}$ and G are canonically isomorphic and so G has dimension d.

For the last statement of the proposition, assume that $\mathcal{P}$ is nilpotent, and so $V^N(P) \subseteq z(\sigma^{N*}P)$ for some N. Since σ is z-linear, it follows that for all $n \geq 1$, we have $V^{nN}(P) \subseteq z^n(\sigma^{nN*}P)$ and therefore, $V : P/z^n \to \sigma^*(P/z^n)$ is nilpotent, which implies that G_n is infinitesimal. If $\mathcal{P}$ is étale, then $V : P \to \sigma^* P$ and so all $V : P/z^n \to \sigma^*(P/z^n)$ are isomorphisms, and so G_n are étale. □

Theorem 3.57. *Let $\mathcal{O}$ be the ring of integers of a non-Archimedean local field of positive characteristic. Let S be an $\mathcal{O}$-scheme and assume that the image of the maximal ideal of $\mathcal{O}$ in $\mathcal{O}_S$ is Zariski locally nilpotent. Then:*

(a) *The functors $\mathcal{P} \mapsto DA(\mathcal{P})$ and $G \mapsto \mathcal{D}(G)$, constructed above, between the category of Drinfeld displays over S and the category of z-divisible local Anderson modules over S are quasi-inverses to each other.*

(b) *The functor DA commutes with base change.*

(c) *$\mathcal{P}$ is nilpotent (respectively étale) if and only if $DA(\mathcal{P})$ is infinitesimal (respectively étale).*

(d) *The tangent space of a Drinfeld display $\mathcal{P}$ is canonically isomorphic, as an $\mathcal{O}_S$-module, to the tangent space of $DA(\mathcal{P})$.*

(e) *The rank and dimension of a Drinfeld display $\mathcal{P}$ are respectively equal to the height and dimension of $DA(\mathcal{P})$.*

(f) *For all n, let $\mathcal{P}_n$ be the z-sheaf $\mathcal{P}/z^n$. Then, there is a canonical isomorphism*

$$\mathcal{G}(\mathcal{P}_n) \cong DA(\mathcal{P})_n.$$

(g) *Let S be the spectrum of a perfect field k, and $\mathcal{P} = (P, V)$ a Drinfeld display over S. Then P is canonically isomorphic, as a $k[\![z]\!]$-module, to the ramified Dieudonné module $\mathbb{H}(DA(\mathcal{P}))$.*

Proof. (a) In fact we showed in the proof of Proposition 3.56 that $\mathcal{D}\big(DA(\mathcal{P})\big)$ is canonically isomorphic to $\mathcal{P}$. The other direction also follow immediately from the constructions of these functors: take a z-divisible local Anderson module G, then $\mathcal{D}(G)$ is the inverse limit of $\mathcal{D}(G_n)$ and we have showed that $\mathcal{D}(G_n)$ is $\mathcal{D}(G)/z^n$ (see Remark 3.54). We deduce that

$$DA\big(\mathcal{D}(G)\big) \cong \varinjlim_n \mathcal{G}(\mathcal{D}(G)/z^n) \cong \varinjlim_n \mathcal{G}\big(\mathcal{D}(G_n)\big) \cong \varinjlim_n G_n = G$$

(b) It follows from the definition of base change of Drinfeld displays and $DA(\mathcal{P})_n$ that for any S-scheme T, we have

$$DA(\mathcal{P})_{n,T} \cong DA(\mathcal{P}_T)_n$$

This proves (b).

(c) This follows from Propositions 3.47 and 3.56.

(d) This follows from (a) and Remark 3.54 1).

(e) This follows from (a) and Proposition 3.47.

(f) This is just the definition of $DA(\mathcal{P})$.

(g) This follows from the construction of DA and Remark 3.8 3).

□

Corollary 3.58. *Let G be a (strict) z-divisible local Anderson module. The finite φ-modules G_n have a natural structure of a (strict) finite v-module.*

Proof. Let $\mathcal{P} = (P, V)$ be the Drinfeld display attached to G. By Remark 3.38, there is a morphism $F : \sigma^* P \to P$ and an $N \geq 1$, such that the compositions $F \circ V$ and $V \circ F$ are equal to the multiplication by $(z - \zeta)^N$. Modulo z^n, this morphism induces a morphism $F : \sigma^* P/z^n \to P/z^n$ satisfying the same identity $F \circ V = (z - \zeta)^N = V \circ F$. Therefore the locally free sheaf P/z^n has a natural structure of a Frob-sheaf. It now follows from Theorem 3.57 (f) and Proposition 3.18 that G_n has a natural structure of a finite v-module. If G is strict, then $N = 1$, and so G_n is struct as well. □

3.3 Descent

In this section we show faithfully flat descent for Drinfeld displays. We do this in details for faithfully flat morphisms of $\mathcal{O}$-algebras. The generalization to faithfully flat quasi-compact morphisms $\mathcal{O}$-schemes is straightforward and is left to the reader. Fix a faithfully flat $\mathcal{O}$-algebra homomorphism $A \to B$. At first, we show that descent data on finitely generated projective $B[\![z]\!]$-modules, relative to $A \to B$ are effective (Proposition 3.79). We do this by reducing the question to the usual faithfully flat descent of finitely generated projective B-modules, relative to $A \to B$. For faithfully flat descent of modules and quasi-coherent sheaves we refer to [6], Exp. VIII, Théorème 1.1, Proposition 1.10, Corollaire 1.11, and [12], Ch. II, Théorème 3.2). The point here is that when $A \to B$ is faithfully flat, the induced morphism $A[\![z]\!] \to B[\![z]\!]$ is topologically faithfully flat, and the descent theory for these morphisms exist in the topological setting (i.e., one has to take completed tensor products etc.). However, this is not desirable for our purpose, as the base change of Drinfeld displays are not taken in the topological setting. It seems that a direct (but perhaps longer) way of developing a descent theory, without reducing to the usual case, is possible, but we decided to use the shortcut.

We have the following exact sequence of descent:

$$0 \to A \to B \rightrightarrows B \otimes_A B \mathrel{\substack{\to\\\to\\\to}} B \otimes_A B \otimes_A B \cdots \tag{3.60}$$

Since $\mathbb{Z}[\![z]\!]$ is a topologically faithfully flat $\mathbb{Z}$-algebra, and for any ring R, there is a canonical isomorphism $R[\![z]\!] \cong \mathbb{Z}[\![z]\!] \widehat{\otimes}_{\mathbb{Z}} R$ (the ring R having the discrete topology), this short exact sequence induces the following short exact sequence

$$0 \to A[\![z]\!] \to B[\![z]\!] \rightrightarrows (B \otimes_A B)[\![z]\!] \mathrel{\substack{\to\\\to\\\to}} (B \otimes_A B \otimes_A B)[\![z]\!] \cdots \tag{3.62}$$

Lemma 3.63. *Let M be a flat $A[\![z]\!]$-module. Then the following sequence, induced by sequence (3.62) is exact*

$$0 \to M \to B[\![z]\!] \otimes_{A[\![z]\!]} M \rightrightarrows (B \otimes_A B)[\![z]\!] \otimes_{A[\![z]\!]} M \overset{\rightarrow}{\rightrightarrows} (B^{\otimes 3})[\![z]\!] \otimes_{A[\![z]\!]} M \cdots$$

Proof. Since M is a direct limit of free modules, we are reduced to the case $M = A[\![z]\!]$, and so we have sequence (3.62). □

Theorem 3.64 (Descent for Morphisms)**.** *Let $A \to B$ be a faithfully flat $\mathcal{O}$-algebra homomorphism. Let $\mathcal{P}$ and $\mathcal{P}'$ be Drinfeld displays over A. Then the following sequence is exact:*

$$0 \to \operatorname{Hom}(\mathcal{P}, \mathcal{P}') \to \operatorname{Hom}(\mathcal{P}_B, \mathcal{P}'_B) \rightrightarrows \operatorname{Hom}(\mathcal{P}_{B\otimes_A B}, \mathcal{P}'_{B\otimes_A B}).$$

Proof. Write $\mathcal{P} = (P, V)$ and $\mathcal{P}' = (P', V')$. It follows from Lemma 3.63 that for any projective $A[\![z]\!]$-modules M and N, the sequence

$$0 \to \operatorname{Hom}(M, N) \to \operatorname{Hom}(M_{B[\![z]\!]}, N_{B[\![z]\!]}) \rightrightarrows \operatorname{Hom}(M_{B^{\otimes 2}[\![z]\!]}, N_{B^{\otimes 2}[\![z]\!]}) \quad (3.66)$$

is exact (here, for an A-algebra R and an $A[\![z]\!]$-module P, we denote by $P_{R[\![z]\!]}$ the base change $R[\![z]\!] \otimes_{A[\![z]\!]} P$). Therefore, we only need to show that if $u : P \to P'$ is a morphism such that the base change u_B is in $\operatorname{Hom}(\mathcal{P}_B, \mathcal{P}'_B)$ then it is morphism of Drinfeld displays, i.e., $\sigma^* u \circ V = V' \circ u$. For such a u, the base change $(\sigma^* u \circ V - V' \circ u)_B = \sigma^*(u_B) \circ V - V' \circ u_B$ is zero (as u_B is morphism of Drinfeld displays), and since

$$\operatorname{Hom}(P, \sigma^* P') \to \operatorname{Hom}(P_B, \sigma^* P'_B)$$

is injective, we have that $\sigma^* u \circ V - V' \circ u = 0$ and this finishes the proof. □

Before proceeding with the descent, let us collect some general well-known results regarding projective modules:

Lemma 3.67. *Let R be a ring, I an ideal of R and assume that R is complete in the I-adic topology. Let P be a finitely generated projective R-module. Then P is I-adically complete as well.*[3]

Proof. There exists a finite free R-module F and a surjection $\pi : F \twoheadrightarrow P$. Since P is projective, there exists also a section $s : P \hookrightarrow F$ of π. This implies that the natural morphism

$$\varprojlim_n F/I^n F \to \varprojlim_n P/I^n P$$

[3]Note that if R is Noetherian, then this is true for every finitely generated R-module, as R is itself I-adically complete.

induced by π is surjective and the natural morphism

$$\varprojlim_n P/I^nP \to \varprojlim_n F/I^nF$$

induced by s is injective. As F is finite free and therefore complete, we conclude that the natural morphism $P \to \varprojlim_n P/I^nP$ is a bijection and so, P is complete. □

Lemma 3.68. *Let R be a ring and I an ideal of R. Assume that R is complete for the I-adic topology. Then any idempotent of the matrix ring $\mathbb{M}_m(R/I)$ lifts to an idempotent of $\mathbb{M}_m(R)$.*

Proof. Since R is I-adically complete and $\mathbb{M}_m(R)$ is finite free as an R-module, it is also I-adically complete, which is to say that it is complete for the $I\mathbb{M}_m(R)$-adic topology. The ideal $I\mathbb{M}_m(R)$ being the kernel of the projection $\mathbb{M}_m(R) \twoheadrightarrow \mathbb{M}_m(R/I)$, the result follows from [1], Ch. III, Proposition 2.10, which states the following:

If R' is a (possibly non-commutative) ring, complete for the J-adic topology, where J is a two-sided ideal, then any idempotent of the quotient R'/J lifts to an idempotent of R'. □

Corollary 3.69. *Let R be a ring and I an ideal of R. Assume that R is complete for the I-adic topology. Let $\bar{P}$ a finitely generated projective R/I-module, then $\bar{P}$ lifts, up to isomorphism, to a unique finitely generated projective R-module (i.e., there exists a finitely generated projective R-module P and an isomorphism $P/IP \cong \bar{P}$).*

Proof. Giving a finitely generated projective module over a ring R is the same as giving an idempotent of the matrix algebra $\mathbb{M}_m(R/I)$ for certain $m \geq 1$. So, lifting $\bar{P}$ to a finitely generated projective R-module is the same as lifting idempotents from $\mathbb{M}_m(R/I)$ to $\mathbb{M}_m(R)$, which can be done thanks to Lemma 3.68. The uniqueness follows from Nakayama's lemma. □

Let us now continue with the descent. Let N be a $B[\![z]\!]$-module. We would like to have a descent datum with respect to the sequence (3.62). Note that this is not the usual descent sequence with respect to the homomorphism $A[\![z]\!] \to B[\![z]\!]$.

In order to avoid heavy notations, when confusion is not likely, we denote the $A[\![z]\!]$-algebra homomorphisms (fixing z) induced from A-algebra homomorphisms by the same letter. Let us define some A-algebra homomorphisms that we will need:

- $p_1 : B \to B \otimes_A B, \quad b \mapsto b \otimes 1$ and $p_2 : B \to B \otimes_A B, \quad b \mapsto 1 \otimes b$.
- For every $i \neq j \in \{1, 2, 3\}$, let $p_{ij} : B \otimes_A B \to B \otimes_A B \otimes_A B$ be the A-algebra homomorphism sending the first and respectively second factor of B identically to the i-th and respectively j-th factor of B (e.g., $p_{12}(b_1 \otimes b_2) = b_1 \otimes b_2 \otimes 1$).

  For $n \geq 1$:
- for all $0 \leq i \leq n$, we set $\tilde{\partial}_i : B^{\otimes n} \to B^{\otimes(n+1)}, \quad b_1 \otimes \cdots \otimes b_n \mapsto b_1 \otimes \cdots \otimes b_i \otimes 1 \otimes b_{i+1} \otimes \ldots b_n$.
- $\iota : B \to B^{\otimes n}, \quad b \mapsto 1 \otimes 1 \otimes \cdots \otimes 1 \otimes b$.

We then have the following commutative diagram:

$$\begin{CD} B @>{p_1}>> B^{\otimes 2} \\ @V{\iota}VV @VV{\tilde{\partial}_0^{n-1}}V \\ B^{\otimes n} @>>{\tilde{\partial}_n}> B^{\otimes(n+1)} \end{CD} \tag{3.71}$$

For any ring homomorphism $\psi : R[\![z]\!] \to R'[\![z]\!]$ and any $R[\![z]\!]$-module M, we denote by $\psi^* M$ the base change $R'[\![z]\!] \otimes_{\psi, R[\![z]\!]} M$.

Definition 3.72. (i) A *z-descent datum* relative to the faithfully flat homomorphism $A \to B$ is a pair (N, θ), where N is a $B[\![z]\!]$-module and $\theta : p_1^* N \to p_2^* N$ is an isomorphism satisfying the *cocyle condition*: the following diagram is commutative

$$\begin{CD} p_{12}^* p_1^* N @>{p_{12}^* \theta}>> p_{12}^* p_2^* N \\ @| @| \\ p_{13}^* p_1^* N @>>> p_{23}^* p_1^* N \\ @V{p_{12}^* \theta}VV @VV{p_{12}^* \theta}V \\ p_{13}^* p_2^* N @= p_{23}^* p_2^* N \end{CD} \tag{3.74}$$

(ii) Morphisms of z-descent data from (N, θ) to (N', θ') are $B[\![z]\!]$-linear morphisms from N to N' that commute with θ and θ'. ▲

Remark 3.75. The composition of p_1 and p_2 with $A[\![z]\!] \to B[\![z]\!]$ are equal and so, for any $A[\![z]\!]$-module M, the base changes $p_1^*\big(B[\![z]\!] \otimes_{A[\![z]\!]} M\big)$ and $p_2^*\big(B[\![z]\!] \otimes_{A[\![z]\!]} M\big)$ are canonically isomorphism (with the canonical isomorphism denoted by *can.*) and therefore, we have a z-descent datum $(B[\![z]\!] \otimes_{A[\![z]\!]} M, can.)$. ◊

Definition 3.76. A z-descent datum (N,θ) relative to $A \to B$ is called *effective* if it is isomorphic to $(B[\![z]\!] \otimes_{A[\![z]\!]} M, can.)$ for some $A[\![z]\!]$-module M. ▲

Now let (N,θ) be a z-descent datum relative to $A \to B$. The isomorphism θ defines a homomorphism

$$\theta^\sharp : N \to (B \otimes_A B)[\![z]\!] \otimes_{\iota,B[\![z]\!]} N, \quad \theta^\sharp(x) = \theta(1 \otimes x)$$

It satisfies

$$\forall g \in B[\![z]\!],\ x \in N : \quad \theta^\sharp(gx) = p_1(g)\theta^\sharp(x) \tag{3.78}$$

For all $n \geq 1$ and $0 \leq i \leq n$, we are going to define $B[\![z]\!]$-linear homomorphisms

$$\partial_i : B^{\otimes n}[\![z]\!] \otimes_{\iota,B[\![z]\!]} N \to B^{\otimes(n+1)}[\![z]\!] \otimes_{\iota,B[\![z]\!]} N$$

- for $i < n$, set $\partial_i = \tilde{\partial}_i \otimes \mathrm{Id}_N$
- $\partial_n = \tilde{\partial}_n \otimes \theta^\sharp$

Note that because of the commutative diagram (3.71) and identity (3.78), ∂_n is well-defined. Now, for all $n \geq 1$, we set:

$$\delta_n := \sum_{i=0}^{n} (-1)^i \partial_i : B^{\otimes n}[\![z]\!] \otimes_{\iota,B[\![z]\!]} N \to B^{\otimes(n+1)}[\![z]\!] \otimes_{\iota,B[\![z]\!]} N$$

Proposition 3.79. *Let $A \to B$ be a faithfully flat homomorphism and (N,θ) a z-descent datum relative to $A \to B$. Assume that N is a finitely generated projective $B[\![z]\!]$-module. Then:*

(a) the sequence

$$N \xrightarrow{\delta_1} (B^{\otimes 2})[\![z]\!] \otimes_{\iota,B[\![z]\!]} N \xrightarrow{\delta_2} (B^{\otimes 3})[\![z]\!] \otimes_{\iota,B[\![z]\!]} N \xrightarrow{\delta_3} \cdots \tag{3.81}$$

is exact.

(b) $N_0 := \mathrm{Ker}(\delta_1)$ is a finitely generated projective $A[\![z]\!]$-module.

(c) the canonical $B[\![z]\!]$-linear homomorphism $B[\![z]\!] \otimes_{A[\![z]\!]} N_0 \to N$ is an isomorphism, and so (N,θ) is effective.

Proof. To avoid heavy notations, in this proof, unless otherwise specified, tensor products are $\otimes_{\iota,B[\![z]\!]}$.

(a) The cocycle condition (commutative diagram 3.72) implies that the sequence (3.81) is a complex (i.e., $\delta_{n+1} \circ \delta_n = 0$). For every $i \geq 0$, the complex (3.81) induces a complex

$$z^i N \xrightarrow{\delta_1} z^i (B \otimes_A B)[\![z]\!] \otimes N \xrightarrow{\delta_2} z^i (B \otimes_A B \otimes_A B)[\![z]\!] \otimes N \xrightarrow{\delta_3} \cdots$$

In other words, we obtain a complex, where each term is the filtration by $z^i B^{\otimes n}[\![z]\!] \otimes N$. Since N is projective, the i-th graded object is

$$z^i N/z^{i+1}N \xrightarrow{\delta_1} (z^i B^{\otimes 2}[\![z]\!]/z^{i+1}B^{\otimes 2}[\![z]\!]) \otimes N \xrightarrow{\delta_2} \cdots$$

$$\cdots \xrightarrow{\delta_{n-1}} (z^i B^{\otimes n}[\![z]\!]/z^{i+1}B^{\otimes n}[\![z]\!]) \otimes N \xrightarrow{\delta_n} \cdots$$

But, for each n, we have canonical isomorphisms

$$(z^i B^{\otimes n}[\![z]\!]/z^{i+1}B^{\otimes n}[\![z]\!]) \otimes N \cong (B^{\otimes n}[\![z]\!]/zB^{\otimes n}[\![z]\!]) \otimes N \cong$$

$$B^{\otimes n} \otimes_{\iota,B} N/zN.$$

Therefore, the last complex is the usual descent sequence of the B-module N/zN relative to the faithfully flat morphism $A \to B$, which is exact, by usual descent (note that the z-descent datum (N,θ) induces a usual descent datum $(N/zN,\bar{\theta})$ on the B-module N/zN).

Now, using the long exact sequence of cohomology, we conclude that, for all $i \geq 1$, the complex

$$N/z^iN \xrightarrow{\delta_1} (B^{\otimes 2}[\![z]\!] \otimes N)/(z^i B^{\otimes 2}[\![z]\!] \otimes N) \xrightarrow{\delta_2} \cdots$$

$$\cdots \xrightarrow{\delta_{n-1}} (B^{\otimes n}[\![z]\!] \otimes N)/(z^i B^{\otimes n}[\![z]\!] \otimes N) \xrightarrow{\delta_n} \cdots$$

is exact as well. Since the terms of this projective system of complexes satisfy the Mittag-Leffler condition (the transition morphisms are sujective), its projective limit remains exact. By Lemma 3.67, we have a canonical isomorphism

$$\varprojlim_i (B^{\otimes n}[\![z]\!] \otimes N)/(z^i B^{\otimes n}[\![z]\!] \otimes N) \cong B^{\otimes n}[\![z]\!] \otimes N$$

and therefore, the complex (3.81) is exact. This proves (a).

(b) Consider the following exact sequence, which is the usual sequence attached to the descent datum $(N/zN,\bar{\theta})$ relative to $A \to B$

$$0 \to \mathrm{Ker}(\bar{\delta}_1) \to N/zN \xrightarrow{\bar{\delta}_1} (B \otimes_A B) \otimes_{\iota,B} N/zN \xrightarrow{\bar{\delta}_2} \cdots$$

Let us denote $\mathrm{Ker}(\bar{\delta}_1)$ by $\bar{N}_0$. Note that the projection $N \twoheadrightarrow N/zN$ induces a surjective morphism $N_0 \twoheadrightarrow \bar{N}_0$. By the usual descent, $\bar{N}_0$ is a finite generated projective A-module and we have a canonical isomorphism

$$B \otimes_A \bar{N}_0 \cong N/zN$$

By Corollary 3.69, there is a finitely generated projective $A\llbracket z\rrbracket$-module M and an isomorphism $M/zM \cong \bar{N}_0$. Since M is projective and $N_0 \twoheadrightarrow \bar{N}_0$ is surjective, we obtain a morphism $M \xrightarrow{\rho} N_0$ that makes the following triangle commutative:

$$\begin{array}{ccc} M & \xrightarrow{\quad\rho\quad} & N_0 \\ & \searrow\!\!\!\!\searrow \quad \swarrow\!\!\!\!\swarrow & \\ & M/zM \cong \bar{N}_0 & \end{array}$$

We are going to show that ρ is an isomorphism. This will prove (b), as M is finitely generate projective. The composition $M \xrightarrow{\rho} N_0 \hookrightarrow N$ induces a map $B\llbracket z\rrbracket \otimes_{A\llbracket z\rrbracket} M \xrightarrow{\rho^\sharp} N$ that makes the following diagram commutative:

$$\begin{array}{ccc} B\llbracket z\rrbracket \otimes_{A\llbracket z\rrbracket} M & \xrightarrow{\rho^\sharp} & N \\ \downarrow & & \downarrow \\ B \otimes_A \bar{N}_0 & \longrightarrow & N/zN \end{array}$$

where the right vertical morphism is the natural projection, the bottom horizontal morphism is the B-linearization of the the injection $\bar{N}_0 \hookrightarrow N/zN$ and the left vertical morphism is the reduction mod z morphism, noting that

$$(B\llbracket z\rrbracket \otimes_{A\llbracket z\rrbracket} M)/(zB\llbracket z\rrbracket \otimes_{A\llbracket z\rrbracket} M) \cong B \otimes_A (M/zM) \cong B \otimes_A \bar{N}_0.$$

As we said above, the bottom map is an isomorphism. So, by Nakayama's lemma, the top morphism $\rho^\sharp$ is an isomorphism as well. It now follows from Lemma 3.63 that $M \xrightarrow{\rho} N_0$ is an isomorphism $\big($note that the equalizer of $B\llbracket z\rrbracket \otimes_{A\llbracket z\rrbracket} M \rightrightarrows (B \otimes_A B)\llbracket z\rrbracket \otimes_{A\llbracket z\rrbracket} M$, after the identification $B\llbracket z\rrbracket \otimes_{A\llbracket z\rrbracket} M \cong N$ is just the kernel of $N \xrightarrow{\delta_1} (B \otimes_A B)\llbracket z\rrbracket \otimes_{\iota, B\llbracket z\rrbracket} N\big)$.

(c) The composition

$$B\llbracket z\rrbracket \otimes_{A\llbracket z\rrbracket} M \xrightarrow{\mathrm{Id} \otimes \rho} B\llbracket z\rrbracket \otimes_{A\llbracket z\rrbracket} N_0 \to N$$

is $\rho^\sharp$. In the proof of (b), we showed that it is an isomorphism. We also proved that ρ is an isomorphism. Thus, the morphism $B\llbracket z\rrbracket \otimes_{A\llbracket z\rrbracket} N_0 \to N$ is an isomorphism as well.

□

Corollary 3.82. *The functor from the category of finitely generated projective $A[\![z]\!]$-modules to the category of z-descent data on a finitely generated projective $B[\![z]\!]$-module, relative to $A \to B$, that sends M to $(B[\![z]\!] \otimes_{A[\![z]\!]} M, can.)$ is an equivalence of categories.*

Proof. This is part (c) of the proposition. Note that part (b) of the proposition gives a quasi-inverse of this functor. □

Now, we want to define descent data for Drinfeld displays and show that they are effective as well. Recall that for a ring homomorphism $a : R \to R'$ and a Drinfeld display $\mathcal{P}$ over R, we denote by $a^*\mathcal{P}$ the base change of $\mathcal{P}$ along a (cf. Remark 3.38).

Definition 3.83. Let $A \to B$ a faithfully flat $\mathcal{O}$-algebra homomorphism.

(i) Let $\mathcal{P}$ be a Drinfeld display over B. A *descent datum* on $\mathcal{P}$ relative to $A \to B$ is an isomorphism of Drinfeld displays $\theta : p_1^*\mathcal{P} \to p_2^*\mathcal{P}$ satisfying the *cocyle condition*: the following diagram is commutative

$$\begin{array}{ccc} p_{12}^*p_1^*\mathcal{P} & \xrightarrow{p_{12}^*\theta} & p_{12}^*p_2^*\mathcal{P} \\ \| & & \| \\ p_{13}^*p_1^*\mathcal{P} & \longrightarrow & p_{23}^*p_1^*\mathcal{P} \\ {\scriptstyle p_{12}^*\theta}\downarrow & & \downarrow{\scriptstyle p_{12}^*\theta} \\ p_{13}^*p_2^*\mathcal{P} & = & p_{23}^*p_2^*\mathcal{P} \end{array} \tag{3.85}$$

(ii) If $(\mathcal{P}, \theta)$ and $(\mathcal{P}', \theta')$ are Drinfeld displays together with descent data, a morphisms from $(\mathcal{P}, \theta)$ to $(\mathcal{P}', \theta')$ are morphisms from $\mathcal{P}$ to $\mathcal{P}'$ that commute with θ and θ'.

(ii) We call a Drinfeld display with descent datum *effective*, if it "comes" from a Drinfeld display over A (cf. Remark 3.75 and Definition 3.76). ▲

Theorem 3.86. *(Descent for Objects) Let $A \to B$ be a faithfully flat $\mathcal{O}$-algebra homomorphism. The functor from the category of Drinfeld displays over A to the category of Drinfeld displays over B with descent data, that sends $\mathcal{P}$ to $(\mathcal{P}_B, can.)$ is an equivalence of categories. The same holds for nilpotent Drinfeld displays and strict Drinfeld displays.*

Proof. Let $(\mathcal{P}, \theta)$ be a Drinfeld display with descent datum. Write $\mathcal{P} = (P, V)$. Forgetting V, the isomorphism θ is just a descent datum on the finitely generated projective $B[\![z]\!]$-module P, and so by Proposition 3.79, the map $B[\![z]\!] \otimes_{A[\![z]\!]} M \to P$ is an isomorphism, where M is the kernel of

$$\delta_1 : P \to (B \otimes B)[\![z]\!] \otimes_{\iota, B[\![z]\!]} P.$$

If we show that the restriction of $V : P \to \sigma^*P$ to M induces a map $V : M \to \sigma^*M$, then we are done, as $\mathcal{P} = (P, V)$ would be the base change of $\mathcal{P}_0 := (M, V_{|M})$. As we saw in the proof of Lemma 3.64, the sequence (3.66)

$$0 \to \operatorname{Hom}(M, \sigma^*M) \to \\ \operatorname{Hom}(M_{B[\![z]\!]}, \sigma^*M_{B[\![z]\!]}) \rightrightarrows \operatorname{Hom}(M_{B^{\otimes 2}[\![z]\!]}, \sigma^*M_{B^{\otimes 2}[\![z]\!]})$$

is exact. Since $\theta : p_1^*\mathcal{P} \to p_2^*\mathcal{P}$ is an isomorphism of Drinfeld displays, after the identifications $M_{B[\![z]\!]} \cong P$ and $\sigma^*M_{B[\![z]\!]} \cong \sigma^*P$, we see that $V : M_{B[\![z]\!]} \to \sigma^*M_{B[\![z]\!]}$ is in the equalizer of

$$\operatorname{Hom}(M_{B[\![z]\!]}, \sigma^*M_{B[\![z]\!]}) \rightrightarrows \operatorname{Hom}(M_{(B\otimes_A B)[\![z]\!]}, \sigma^*M_{(B\otimes_A B)[\![z]\!]})$$

and so comes from a morphism $M \to \sigma^*M$. This finishes the proof of the first statement.

Now assume that $\mathcal{P} = (P, V)$ is nilpotent, which means there is some $m \geq 1$, such that $P \xrightarrow{V^N} \sigma^{m*}P/z$ is trivial. We want to show that $\mathcal{P}_0 = (M, V_{|M})$ is nilpotent as well. This follows from the facts that the diagram

$$\begin{array}{ccc} M & \hookrightarrow & P \\ {\scriptstyle V^m}\downarrow & & \downarrow{\scriptstyle V^m} \\ \sigma^{m*}M/zM & \longrightarrow & \sigma^{m*}P/z \end{array}$$

is commutative and that the bottom map $\sigma^{m*}M/zM \to \sigma^{m*}P/z$ is injective (note that $\sigma^{m*}P/z \cong B \otimes_A \sigma^{m*}M/z$, because $P \cong B[\![z]\!] \otimes_{A[\![z]\!]} M$).

Finally, we want to show that if $\mathcal{P}_0$ is a Drinfeld display over A, such that its base change $\mathcal{P}$ to B is strict, it is strict as well. Let C be the tangent space of $\mathcal{P}_0$. Then the tangent space of $\mathcal{P}$ is canonically isomorphic to $B \otimes_A C$. Let $m \geq 1$ be such that $(z-\zeta)^m$ kills $B \otimes_A C$. Then, since $A \to B$ is faithfully flat, $(z-\zeta)^m$ kills C as well. When $m = 1$, this is what we desired.

Note that for the last statement, we could have used the descent for morphisms to descend the morphism $F : \sigma^*P \to P$, with $F \circ V = V \circ F = (z-\zeta)^m$ to a morphism $\sigma^*M \to M$ (as we did for V), which would then automatically satisfy $F \circ V = V \circ F = (z-\zeta)^m$, because of the faithfully flatness (cf. Remark 3.38). □

We can generalize everything that we did so far to faithfully flat and quasi-compact morphisms of $\mathcal{O}$-schemes. Indeed, since Drinfeld displays over general

$\mathcal{O}$-schemes are Zariski locally given by free sheaves, and since there exists fpqc descent for sheaves and locally free sheaves ([6], Exp. VIII, Théorème 1.1, Proposition 1.10), we could either use the above results and "glue" objects over affine bases to obtain the general results, or do what we did verbatim, with A and respectively B replaced with $\mathcal{O}_S$ and respectively $\mathcal{O}_T$ (here the faithfully flat morphism $A \to B$ is being replaced with an fpqc morphism $T \to S$). We can therefore obtain the following theorem, whose straightforward proof is omitted and left to the reader.

Theorem 3.87. *Let $T \to S$ be a faithfully flat and quasi-compact morphism of $\mathcal{O}$-schemes. Then:*

(a) (Descent for Morphisms) Let $\mathcal{P}$ and $\mathcal{P}'$ be Drinfeld displays over S. Then the following sequence is exact:

$$0 \to \mathrm{Hom}(\mathcal{P}, \mathcal{P}') \to \mathrm{Hom}(\mathcal{P}_T, \mathcal{P}'_T) \rightrightarrows \mathrm{Hom}(\mathcal{P}_{T\times_S T}, \mathcal{P}'_{T\times_S T}).$$

(b) (Descent for Objects) The functor from the category of Drinfeld displays over S to the category of Drinfeld displays over T with descent data, that sends $\mathcal{P}$ to $(\mathcal{P}_T, can.)$ is an equivalence of categories. The same holds for nilpotent Drinfeld displays and strict Drinfeld displays.

3.4 Rigidity

In this section, we want to prove a rigidity statement for Drinfeld displays, in the sense of the rigidity for p-divisible groups (cf. [11] Lemma 1.1.3).

Recall that on the our fixed base $\mathcal{O}$-scheme S, element ζ is locally nilpotent (cf. Notations 0.1).

Theorem 3.88. *Let $\mathcal{I}$ be an ideal sheaf of $\mathcal{O}_S$, which is Zariski locally nilpotent. Denote by $\bar{S}$ the closed subscheme of S given by $\mathcal{I}$. Let $\mathcal{P}, \mathcal{P}'$ be Drinfeld displays over S. Then the reduction modulo $\mathcal{I}$ homomorphism*

$$\mathrm{Hom}_S(\mathcal{P}, \mathcal{P}') \to \mathrm{Hom}_{\bar{S}}(\mathcal{P}_{\bar{S}}, \mathcal{P}'_{\bar{S}})$$

is injective. Moreover, its cokernel is annihilated by z^M, where $M : S \to \mathbb{N}$ is a locally constant function that depends on the order of nilpotence of $\mathcal{I}$ and $\zeta \in \mathcal{O}_S$ and the degree of nilpotence of $\mathcal{P}'$ (cf. Remark 3.38).

Proof. We can assume that S is affine, say $S = \mathbf{Spec}(A)$, and so $\mathcal{I} = \widetilde{I}$ for some ideal I of A. Write $\mathcal{P} = (P, V)$ and $\mathcal{P}' = (P', V)$. By localizing A if necessary, we can assume that there are natural numbers N, n and d, with $q^n \geq d$ such that $I^{q^N} = 0, \zeta^{q^n} = 0$ and $(z - \zeta)^d \cdot \mathrm{Coker}(V : P' \to \sigma^* P') = 0$.

So, $\sigma^N(I[\![z]\!]) = 0$, where, as usual, $\sigma : A[\![z]\!] \to A[\![z]\!]$ is the $\mathbb{F}_q$-linear Frobenius fixing z, and we denote by $I[\![z]\!]$, the ideal of $A[\![z]\!]$, consisting of power series with coefficients in I. We also have $z^{q^n} \cdot \operatorname{Coker}(V : P' \to \sigma^* P') = 0$ and so, $z^{q^n}\sigma^* P' \subseteq V(P')$ (note that $(z-\zeta)^d$ annihilates the tangent space of $\mathcal{P}'$ and we have $z^{q^n} = (z-\zeta)^{q^n}$).

Before we continue, recall that by V^m we mean $\sigma^{m*}V \circ \cdots \circ \sigma^* V \circ V$ (cf. Definition 3.37).

Let us prove the injectivity. Take an element α in the kernel of

$$\operatorname{Hom}_S(\mathcal{P}, \mathcal{P}') \to \operatorname{Hom}_{\bar{S}}(\mathcal{P}_{\bar{S}}, \mathcal{P}'_{\bar{S}}).$$

This means that $\alpha(P) \subseteq I[\![z]\!]P'$. For all $m \geq 1$, the diagram

$$\begin{array}{ccc} P & \xrightarrow{\ \alpha\ } & P' \\ {\scriptstyle V^m}\downarrow & & \downarrow{\scriptstyle V^m} \\ \sigma^{m*}P & \xrightarrow[\operatorname{Id}\otimes_{\sigma^{m*},A[\![z]\!]}\alpha]{} & \sigma^{m*}P' \end{array}$$

is commutative. Since $\alpha(P) \subseteq I[\![z]\!]P'$ and $\sigma^N(I[\![z]\!]) = 0$, the bottom morphism (with $m = N$) is zero, and so $V^N \circ \alpha = 0$. Now, as V is injective, we have $\alpha = 0$ as desired.

Now, let us prove that the cokernel is z-torsion. In fact, we want to show that for every $\bar{\rho} : \mathcal{P}_{\bar{S}} \to \mathcal{P}'_{\bar{S}}$, the homomorphism $z^{Nq^n}\bar{\rho}$ lifts to a unique homomorphism $\rho : \mathcal{P} \to \mathcal{P}'$. The uniqueness, of course, follows from the previous statement.

Since P is a projective $A[\![z]\!]$-module, and $P' \to P'/I[\![z]\!]P'$ is surjective, there is a lift $\rho : P \to P'$ of $\bar{\rho}$. Set $\rho_0 := z^{Nq^n}\rho$. The factor z^{Nq^n} ensures that $(\sigma^{N*}\rho_0 \circ V^N)(P) \subseteq V^N(\sigma^{N*}P')$, and so,

$$\tilde{\rho} := V^{-N} \circ \sigma^{N*}\rho_0 \circ V^N : P \to P'$$

is a well-defined $A[\![z]\!]$-linear homomorphism. As $\bar{\rho}$ and so its multiple $z^{Nq^n}\bar{\rho}$ too is a morphism of Drinfeld displays over A/I, the reductions of $\tilde{\rho}$ and $V^{-1} \circ \sigma^*\rho_0 \circ V$ modulo $I[\![z]\!]$ are both equal to $z^{Nq^n}\bar{\rho}$. This implies that

$$\rho_0 - V^{-1} \circ \sigma^*\rho_0 \circ V \in I[\![z]\!] \operatorname{Hom}_{A[\![z]\!]}(P, P'). \tag{3.90}$$

Now, we claim that $\tilde{\rho}$ commutes with Verschiebungen and so, is a morphism of Drinfeld displays. This will then finish the proof, as $\tilde{\rho}$ is a lift of $z^{Nq^n}\bar{\rho}$. We need to show that

$$V \circ \tilde{\rho} - \sigma^*\tilde{\rho} \circ V = 0.$$

We have

$$V \circ \tilde{\rho} - \sigma^* \tilde{\rho} \circ V = \sigma^* V^{-(N-1)} \circ \big(\sigma^{N*}(\rho_0 - V^{-1} \circ \sigma^* \rho_0 \circ V)\big) \circ V^N.$$

Since $I^{q^N} = 0$, by (3.90) we have

$$\sigma^{N*}(\rho_0 - V^{-1} \circ \sigma^* \rho_0 \circ V) = 0$$

and so $V \circ \tilde{\rho} - \sigma^* \tilde{\rho} \circ V = 0$ as desired. □

Remark 3.91. In the above proof, we showed that the cokernel of

$$\mathrm{Hom}_S(\mathcal{P}, \mathcal{P}') \to \mathrm{Hom}_{\bar{S}}(\mathcal{P}_{\bar{S}}, \mathcal{P}'_{\bar{S}})$$

is annihilated by z^{Nq^n}, where N, n are such that $\mathcal{I}^{q^N} = 0$, $\zeta^{q^n} = 0$ and the degree of nilpotence of $\mathcal{P}'$ is $\leq q^n$. ◊

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