Exterior powers of π-divisible modules over fields

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Abstract. Let $\mathcal{O}$ be the ring of integers of a non-Archimedean local field and π a fixed uniformizer of $\mathcal{O}$. We prove that the exterior powers of a π-divisible $\mathcal{O}$-module scheme of dimension at most 1 over a field exist and commute with field extensions. We calculate the height and the dimension of the exterior powers in terms of the height of the given π-divisible $\mathcal{O}$-module scheme.

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0 Introduction

In this paper, we define and prove the existence of "exterior powers" of π-divisible modules of dimension at most one over a base field, where π is a uniformizer of a non-Archimedean local field.

Before stating the main results of this paper, let us establish some definitions and notations. Let $\mathcal{O}$ be the ring of integers of a non-Archimedean local field and fix a uniformizer π. Let $\mathcal{M}$ and $\mathcal{N}$ be π-divisible modules over a base scheme S.

Definition 3.1. An *$\mathcal{O}$-alternating morphism* $\varphi : \mathcal{M}^r \to \mathcal{N}$ is a system of S-morphisms $\{\varphi_n : \mathcal{M}_n^r \to \mathcal{N}_n\}_n$ over S, compatible with the projections $\pi. : \mathcal{M}_{n+1} \twoheadrightarrow \mathcal{M}_n$ and $\pi. : \mathcal{N}_{n+1} \twoheadrightarrow \mathcal{N}_n$, where

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$\varphi_n : \mathcal{M}_n^r \to \mathcal{N}_n$ are $\mathcal{O}$-alternating morphism of fppf sheaves of $\mathcal{O}$-modules. Denote the group of $\mathcal{O}$-alternating morphisms from $\mathcal{M}^r$ to $\mathcal{N}$ by $\mathrm{Alt}_S^{\mathcal{O}}(\mathcal{M}^r, \mathcal{N})$.

Definition 3.2. An $\mathcal{O}$-alternating morphism $\lambda : \mathcal{M}^r \to \mathcal{M}'$, or by abuse of terminology, the π-divisible module $\mathcal{M}'$, is called a r^{th} *exterior power* of $\mathcal{M}$ over $\mathcal{O}$, if for all π-divisible module $\mathcal{N}$ over S, the induced morphism

$$\mathrm{Hom}_S^{\mathcal{O}}(\mathcal{M}', \mathcal{N}) \to \mathrm{Alt}_S^{\mathcal{O}}(\mathcal{M}^r, \mathcal{N}), \quad \psi \mapsto \psi \circ \lambda,$$

is an isomorphism. If such $\mathcal{M}'$ and λ exist, we write $\bigwedge^r \mathcal{M}$ for $\mathcal{M}'$.

We can now state the main result of this paper ($p \neq 2$):

Theorem 3.39. *Let $\mathcal{M}$ be a π-divisible module of height h over a base field k. Assume that the dimension of $\mathcal{M}$ is at most 1. Then the j^{th}-exterior power of $\mathcal{M}$ in the category of π-divisible modules over k exists, and has height $\binom{h}{r}$. If the dimension of $\mathcal{M}$ is 1, then $\bigwedge^r \mathcal{M}$ has dimension $\binom{h-1}{r-1}$, otherwise, it has dimension zero. Moreover, for every positive natural number n, we have $(\bigwedge^r \mathcal{M})[\pi^n] \cong \bigwedge^r(\mathcal{M}[\pi^n])$. Furthermore, for any field extension ℓ/k, the base change morphism $\bigwedge^r(\mathcal{M}_\ell) \to (\bigwedge^r \mathcal{M})_\ell$ is an isomorphism.*

The condition on p is for technical reasons, whereas the condition on the dimension is very crucial. This can be explained by looking at the case of p-divisible groups over perfect fields of characteristic p. There, one way of seeing it, is to look at the slopes of the crystal attached to p-divisible groups. Let G be a p-divisible group and let M be its crystal. It is natural to expect that the crystal attached to $\bigwedge^r G$, if it exists, is isomorphic to $\bigwedge^r M$. However, as soon as $r > 1$ and M has a slope larger than $\frac{1}{2}$, $\bigwedge^r M$ will have a slope larger than 1 and thus cannot be the crystal associated with a p-divisible group. This phenomenon is also reflected in the impossibility of defining a Frobenius F on $\bigwedge^r M$ in such a way that $FV = VF = p$ (in fact this is the main issue!). However, when the dimension is one, it is not hard to define a Frobenius on $\bigwedge^r M$, making it a crystal. What is less obvious, and in fact is the content of this paper, is that the p-divisible group attached to $\bigwedge^r M$ is the r^{th} exterior power of G.

The existence of the exterior powers of π-divisible modules will provide a conceptual and precise reason why the exterior powers of the Galois representations attached to π-divisible modules are the Galois representation of a π-divisible module (the same way Anderson introduced t-motives in order to show that the exterior powers of Galois representations attached to Drinfel'd modules come from Drinfel'd modules [1]). The exterior powers are also used in the (local) Langlands program. For example, they are used to study the Lubin-Tate tower in [10] and Rapoport-Zink spaces in [7]. Finally, the existence of these exterior powers, under the condition on the dimension, was conjectured by Lubin (unpublished manuscript) in his study of explicit local class field theory. The results of this paper can be also served as a basis to generalize the main theorem. In fact, the author will build on these results to show, in the sequel, that the exterior powers of π-divisible modules of dimension at most one, over Noetherian base schemes exist.

Let us explain a brief sketch of the proof of the theorem. Let $\mathcal{M}$ be a π-divisible module of height h and dimension 1 over a base field k, and denote by $\mathcal{M}_n$ the kernel of multiplication by π^n. One main idea is that, although not necessary from the definition, we construct the exterior powers of individual $\mathcal{M}_n$ and show that they form a π-divisible module. Using some results of R. Pink (gathered in A), one shows that the exterior powers of finite $\mathcal{O}$-module schemes over k exist (Theorem 2.5). These are profinite $\mathcal{O}$-module schemes. We then show that $\mathcal{M}_n$ sit in an exact sequence (Proposition 2.25)

$$\bigwedge^r \mathcal{M}_{n+m} \xrightarrow{\pi^n} \bigwedge^r \mathcal{M}_{n+m} \to \bigwedge^r \mathcal{M}_m \to 0.$$

So, in order to show that the system $\bigwedge^r \mathcal{M}_n$ form a π-divisible module, using these exact sequences, one has to show that they have the "right" order. In order to calculate their order, we assume that k is a perfect field of characteristic p. Note that if the characteristic of k is different from p, then $\mathcal{M}$ is étale, and using the "étale dictionary", it is rather easy to construct exterior powers (Proposition 3.3). We use Dieudonné theory to calculate the order of $\bigwedge^r \mathcal{M}_n$ in the connected case. We show that the Dieudonné module of $\bigwedge^r \mathcal{M}_n$ is canonically isomorphic to the r^{th} exterior power of the Dieodonné module of $\mathcal{M}_n$

(Corollary 3.21). Here we use the facts that a) $\mathcal{M}_n$ "comes" from a π-divisible module and b) $\mathcal{M}$ has dimension 1. This is the key result of this paper. Once we know this, we can easily compute the order of $\bigwedge^r \mathcal{M}_n$. This will also allow us to calculate the dimension of $\bigwedge^r \mathcal{M}$.

In subsection 1.1 we collect some preliminary definitions and results on multilinear morphisms used in the paper. In subsection 1.2, we study the behavior of the Dieudonné module with respect to multilinear morphisms (Corollary 1.23). This is an adapted version of Pink's multilinear Dieudonné theory. In section 2, we define exterior powers of group schemes and prove some general facts about them. In particular, we prove the aforementioned exact sequences (Proposition 2.25) and the base change properties of exterior powers (Propositions 2.12 and 2.15). We also compute the Dieudonné module of exterior powers of finite p-group schemes (Proposition 2.32). In section 3, we define alternating morphisms and the exterior powers of π-divisible modules. We then show the existence of the exterior powers of étale π-divisible modules over any base scheme (Proposition 3.3). Subsection 3.3 is the heart of the paper. There, we work over a perfect field of characteristic $p > 2$ and we show that, under the condition on the dimension, the exterior powers of a π-divisible module exist, the exterior powers of the Dieudonné module of a π-divisible module have the structure of a Dieudonné module and they are the Diedonné modules of the the exterior powers of the given π-divisible module, and we calculate the height and the dimension of these exterior powers. In subsection 3.4, we combine these results, and show the main theorem of the paper (Theorem 3.39).

Conventions. Throughout the article, unless otherwise specified, rings are commutative with 1. Group schemes are all commutative. An exact sequence of group schemes is an exact sequence of sheaves on the fppf site over the base.

Notations.

- p is a prime number, q is a power of p and $\mathbb{F}_q$ is the finite field with q elements.
- For natural numbers m and n, the binomial coefficient $\binom{n}{m}$ is defined to be zero when $m > n$.
- Take $\underline{x} = (x_i) \in \mathbb{Z}^r$. We denote by $\max \underline{x}$ resp. $\min \underline{x}$ the integer $\max\{x_1, \ldots, x_r\}$ resp. $\min\{x_1, \ldots, x_r\}$.
- $\mathbb{Z}^r_0 := \{\underline{d} = (d_i) \in \mathbb{Z}^r \,|\, \min \underline{d} = 0\}$. Denote by $\mathbb{Z}^r_{0,<M}$ the subset of $\mathbb{Z}^r_0$ consisting of vectors $\underline{d}$ with $\max \underline{d} < M$.
- For integers $a \leq b$, we set $[\![a, b]\!] := [a, b] \cap \mathbb{Z}$.
- If R is a ring and M is an R-module, we denote by $\ell_R(M)$ the length of M over R.
- We denote the kernel of a homomorphism of group functors $\varphi : \mathcal{F} \to \mathcal{G}$, by $\mathcal{F}[\varphi]$.
- Let X be a scheme over a base scheme S. We identify X with the sheaf $\mathrm{Hom}_S(\_, X)$ on fppf site of S.
- Let X be a scheme over a base scheme S and $f : T \to S$ a morphism. We denote by X_T the fiber product $X \times_S T$. If $\mathcal{F}$ is a sheaf on a Grothendieck site over S, we denote by $f^*\mathcal{F}$ the pullback of $\mathcal{F}$ along f. So f^*X and X_T are identified as sheaves.
- Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on a Grothendieck site. We denote by $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ the sheaf hom from $\mathcal{F}$ to $\mathcal{G}$.
- Let E be a field and F a finite extension of E. We denote by $\mathrm{Res}_{F/E}$ the Weil restriction from the category of affine schemes over F to the category of affine schemes over E. This functor commutes with base change and preserves the group objects.
- Let G be a finite flat group scheme over a base scheme S. Denote the Cartier dual $\underline{\mathrm{Hom}}_S(G, \mathbb{G}_{m,S})$ by G^*.
- Let R be a ring. Denote by $W(R)$ the ring of p-typical Witt vectors with coefficients in R.
- Denote by W_S the ring scheme of Witt vectors over a scheme S with Frobenius F and Verschiebung V.
- Denote by W_m the cokernel of the morphism $V^m : W \to W$ and by $W_{m,n}$ the group scheme $W_m[F^n]$.
- Let k be a perfect field of characteristic p. Denote by $\widetilde{W}$ the direct system $W_1 \xrightarrow{v} W_2 \xrightarrow{v} \ldots$, viewed as an ind-object of the category of commutative group schemes over k. Thus, for any commutative group scheme G over k, we have by definition $\mathrm{Hom}(G, \widetilde{W}) = \varinjlim_n \mathrm{Hom}(G, W_n)$.
- For all m and n consider the morphism (of schemes)
$$\tau_{m,n} : W_{m,n} \hookrightarrow W, \quad (x_0, \ldots, x_{m-1}) \mapsto (x_0, \ldots, x_{m-1}, 0, 0, \ldots).$$
Denote by $\widehat{W}$ the formal group scheme $\bigcup_{m,n} \tau_{m,n}(W_{m,n})$. To avoid heavy notations, when confusion is unlikely, we write τ instead of $\tau_{m,n}$. Note that $\widehat{W}$ is sub-ind-object of $\widetilde{W}$.

- Denote by $\mathbb{W}$ the inverse limit, $\varprojlim W_{m,n}$ with transition morphisms the projections $r : W_{m+1,n} \twoheadrightarrow W_{m,n}$ (truncation) and $f : W_{m,n+1} \twoheadrightarrow W_{m,n}$ (Frobenius). Denote by π_n the projection $\mathbb{W} \twoheadrightarrow \varprojlim W_{m,n} = W[F^n]$.
- Let k be a perfect field of characteristic p. We denote by $\mathbb{E}_k$ the Dieudonné ring over k, i.e., the non-commutative polynomial ring in variables F, V and coefficients in $W(k)$ subject to relations $FV = VF = p$, $F\xi = \xi^\sigma F$ and $V\xi^\sigma = \xi V$ for all $\xi \in W(k)$, where ${}^\sigma : W(k) \to W(k)$ is the Frobenius morphism of $W(k)$.
- Let k be a perfect field of characteristic p. We denote by $\widehat{\mathbb{E}}_k$ the (F,V)-adic completion of $\mathbb{E}_k$.
- Let k be a perfect field of characteristic p and G a finite group scheme over k of p-power order. The contravariant Dieudonné module of G, denoted by $D^*(G)$, is the $\mathbb{E}_k$-module $\mathrm{Hom}(G, CW^u) \cong \mathrm{Hom}(G, \widetilde{W})$. The covariant Dieudonné module of G, denoted by $D_*(G)$, is the $\mathbb{E}_k$-module $D^*(G^*)$. If G is local-local, this module is canonically isomorphic to $\mathrm{Hom}(\mathbb{W}, G)$. For details refer to [4].
- For any ring R, denote by $\underline{\Lambda}_R$ the affine group scheme over R, which associates to every R-algebra A, the multiplicative group $1 + t \cdot A[\![t]\!]$ of formal power series in A with constant term 1. Seen as a functor from schemes over R to Abelian groups, $\underline{\Lambda}_R$ is isomorphic to the product $\prod_{\mathbb{N}_+} \mathbb{A}^1_R$ (cf. [2]).
- Set $F(t) := \prod_{p \nmid n} (1-t)^{\frac{\mu(n)}{n}} \in 1 + t \cdot \mathbb{Z}_{(p)}[\![t]\!]$ (cf. [2]).
- The Artin-Hasse exponential is the following morphism (cf. [2])

$$E : W_{\mathbb{Z}_{(p)}} \to \underline{\Lambda}_{\mathbb{Z}_{(p)}}, \qquad \underline{x} \mapsto E(\underline{x}, t) := \prod_{n \in \mathbb{N}} F(x_n \cdot t^{p^n}).$$

1 Multilinear Sheaf Theory

In this section R is a fixed ring. We will define *R-module schemes* and their *multilinear morphisms* and prove some of their properties. In particular, using some results of [6] (see appendix A) we will study the behavior of the covariant Dieudonné functor with respect to multilinear morphisms.

1.1 R-multilinear morphisms

Definition 1.1. A representable fppf-sheaf of R-modules over S will be called an R-module scheme. ▲

Remark 1.2. 1) Let $\mathcal{F}$ and $\mathcal{G}$ be fppf-sheaves of R-modules. We denote by $\mathrm{Hom}^R(\mathcal{F}, \mathcal{G})$ and resp. $\underline{\mathrm{Hom}}^R(\mathcal{F}, \mathcal{G})$ the R-module of R-linear morphisms and resp. the sheaf (of R-modules) of R-linear morphisms.

2) Let M be a finite locally free R-module scheme over S. The Cartier dual of M, i.e., the group scheme $\underline{\mathrm{Hom}}(M, \mathbb{G}_{m,S})$ has a natural R-module scheme structure given by the action of R on M. ◊

Definition 1.3. Let $M_1, \ldots, M_r, M$ and N be presheaves of R-modules on the fppf site of the scheme S.

(i) An *R-multilinear* or simply *multilinear* morphism from the product $\prod_{i=1}^r M_i$ to N is a morphism of presheaves such that for every S-scheme T, the induced morphism $\prod M_i(T) \to N(T)$ is R-multilinear. The R-module of all such morphisms will be denoted by $\mathrm{Mult}^R(\prod_{i=1}^r M_i, N)$.

(ii) An R-multilinear morphism $M^r \to N$ is called *alternating* if for every S-scheme T, the induced morphism $M(T)^r \to N(T)$ is alternating. The R-module of all such alternating multilinear morphisms is denoted by $\mathrm{Alt}^R(M^r, N)$. ▲

There is a weaker notion of multilinearity which will be useful when we want to map to group schemes rather than R-module schemes.

Definition 1.4. Let $M_1 \ldots, M_r$ and M be presheaves of R-modules and G a presheaf of Abelian groups. We denote by $\widetilde{\mathrm{Mult}}^R(\prod_{i=1}^r M_i, G)$ the group of morphisms $\varphi : \prod_{i=1}^r M_i \to G$ which are multilinear, when M_i are regarded as presheaves of Abelian groups and has the following weaker property than R-linearity: for every S-scheme T, every tuple $(m_1, \cdots, m_r) \in \prod_{i=1}^r M_i(T)$, every $a \in R$ and every $i \in \{2, 3, \cdots, r\}$, we have $\varphi(a \cdot m_1, m_2, \cdots, m_r) = \varphi(m_1, \cdots, m_{i-1}, a \cdot m_i, m_{i+1}, \cdots, m_r)$. The elements of $\widetilde{\mathrm{Mult}}^R(\prod_{i=1}^r M_i, G)$ are called *pseudo-R-multilinear*. Similarly, we define *alternating pseudo-R-multilinear morphisms* and denote by $\widetilde{\mathrm{Alt}}^R(M^r, G)$ the group of such morphisms. ▲

Remark 1.5. Note that the group $\widetilde{\mathrm{Mult}}^r(\prod_{i=1}^r M_i, G)$ has a natural structure of R-module through the action of R on one of the factors $M_1, M_2 \dots M_{r-1}$ or M_r, and this is independent of the factor we choose. Similarly, there is a natural R-module structure on the groups $\widetilde{\mathrm{Alt}}^R(M^r, G)$. $\Diamond$

Definition 1.6. Let $M_1, \dots, M_r, M$ and N be presheaves of R-modules and G a presheaf of Abelian groups. Define contravariant functors from the category of R-module schemes over S to the category of R-modules as follows:

(i)
$$T \mapsto \underline{\mathrm{Mult}}^R(M_1 \times \cdots \times M_r, N)(T) := \mathrm{Mult}_T^R(M_{1,T} \times \cdots \times M_{r,T}, N_T)$$
and resp.
$$T \mapsto \widetilde{\underline{\mathrm{Mult}}}^R(M_1 \times \cdots \times M_r, G)(T) := \widetilde{\mathrm{Mult}}_T^R(M_{1,T} \times \cdots \times M_{r,T}, G_T).$$

(ii)
$$T \mapsto \underline{\mathrm{Alt}}^R(M^r, N)(T) := \mathrm{Alt}_T^R(M_T^r, N_T)$$
and resp.
$$T \mapsto \widetilde{\underline{\mathrm{Alt}}}^R(M^r, G)(T) := \widetilde{\mathrm{Alt}}_T^R(M_T^r, G_T)).$$

▲

We now prove a general proposition on multilinear morphisms which will be used throughout the paper.

Proposition 1.7. *Let $M_1, \dots, M_r, M, N_1, \dots, N_s, N$ and P be sheaves of R-modules over S and G a sheaf of Abelian groups over S. We have natural isomorphisms*
$$\mathrm{Mult}^R(M_1 \times \cdots \times M_r, \underline{\mathrm{Mult}}^R(N_1 \times \cdots \times N_s, P)) \cong \mathrm{Mult}^R(M_1 \times \cdots \times M_r \times N_1 \times \cdots \times N_s, P)$$
$$\mathrm{Mult}^R(M_1 \times \cdots \times M_r, \widetilde{\underline{\mathrm{Mult}}}^R(N_1 \times \cdots \times N_s, G)) \cong \widetilde{\mathrm{Mult}}^R(M_1 \times \cdots \times M_r \times N_1 \times \cdots \times N_s, G)$$
$$\mathrm{Alt}^R(N^r, \underline{\mathrm{Hom}}^R(M, P)) \cong \mathrm{Hom}^R(M, \underline{\mathrm{Alt}}^R(N^r, P))$$
and resp.
$$\widetilde{\mathrm{Alt}}^R(N^r, \underline{\mathrm{Hom}}(M, G)) \cong \mathrm{Hom}^R(M, \widetilde{\underline{\mathrm{Alt}}}^R(N^r, G))$$
functorial in all arguments.

Proof. The proof is a standard (multi-) linear algebra argument. $\square$

Proposition 1.8. *Let S be a scheme over $\mathbb{F}_p$ and $G_1, \dots, G_r$ and H be group schemes over S. Let $\varphi : G_1 \times \cdots \times G_r \to H$ be a multilinear morphism over S. We also denote by $\varphi^{(p)}$ the composite $G_1^{(p)} \times \cdots \times G_r^{(p)} \cong (G_1 \times \cdots \times G_r)^{(p)} \xrightarrow{\varphi^{(p)}} H^{(p)}$.*

a) The following diagram is commutative:
$$\begin{tikzcd}
G_1 \times \cdots \times G_r \arrow[r, "\varphi"] \arrow[d, "F_{G_1} \times \cdots \times F_{G_r}"'] & H \arrow[d, "F_H"] \\
G_1^{(p)} \times \cdots \times G_r^{(p)} \arrow[r, "\varphi^{(p)}"'] & H^{(p)}.
\end{tikzcd}$$

b) If G_1 and H are flat, then the following diagram is commutative:
$$\begin{tikzcd}
G_1^{(p)} \times \cdots \times G_r^{(p)} \arrow[r, "\varphi^{(p)}"] & H^{(p)} \arrow[dd, "V_H"] \\
G_1^{(p)} \times G_2 \times \cdots \times G_r \arrow[u, "\mathrm{Id} \times F_{G_2} \times \cdots \times F_{G_r}"] \arrow[d, "V_{G_1} \times \mathrm{Id} \times \cdots \times \mathrm{Id}"'] & \\
G_1 \times \cdots \times G_r \arrow[r, "\varphi"'] & H.
\end{tikzcd}$$

The similar result holds with Verschiebung in G_i for $1 \leq i \leq r$.

Proof. a) This follows from the fact that Frobenius commutes with direct products.

b) When $r = 1$, this is the functoriality of Verschiebung. For $r = 2$, this is proved in [3], Ch.IV, §3, Cor.4.7, p. 516. The general case follows easily.

$\square$

1.2 R-multilinear covariant Dieudonné theory

Fix a perfect field k of characteristic $p > 0$. Unless otherwise specified, all schemes are defined over k.

Remark 1.9. Let M be a finite p-torsion R-module scheme. By functoriality, the Dieudonné of M (covariant or contravariant) has a natural action of $\mathbb{E}_k \otimes_{\mathbb{Z}} R$. $\lozenge$

Definition 1.10. Set $\widetilde{W}^* := \varprojlim_n W_n^* = \varprojlim_n \varprojlim_G G^*$, where G runs through finite subgroups of W_n. $\blacktriangle$

Remark 1.11. Using Proposition A.1, for every finite p-group scheme G over k, there exists a canonical isomorphism $D_*(G) \cong \mathrm{Hom}(\widetilde{W}^*, G)$. If G is unipotent, then the inclusion $\mathrm{Hom}(\mathbb{W}, G) \hookrightarrow \mathrm{Hom}(\widetilde{W}^*, G)$, induced by the canonical projection $\widetilde{W}^* \twoheadrightarrow \mathbb{W}$ is in fact an isomorphism. $\lozenge$

Construction 1.12. Fix an $r \in \mathbb{N}$ and an element $\underline{d} \in \mathbb{Z}_0^r$. For every n, the composition

$$\mathbb{W}^r \xrightarrow{\pi_{n+d_1} \times \cdots \times \pi_{n+d_r}} W[F^{n+d_1}] \times \cdots \times W[F^{n+d_r}] \hookrightarrow W \times \cdots \times W \xrightarrow{\text{mult}} W$$

has image inside the subgroup scheme $W[F^n]$, because $\min \underline{d} = 0$ and Frobenius is a ring homomorphism. Therefore, for every n, we have a multilinear morphism $\zeta_{\underline{d},n} := \mathbb{W}^r \to W[F^n]$ and these morphisms (for all n) are compatible with respect to the projections $F : W[F^{n+1}] \twoheadrightarrow W[F^n]$, and thus, they induce a multilinear morphism $\zeta_{\underline{d}} := \mathbb{W}^r \to \mathbb{W}$ with the property that for all n, $\pi_n \circ \zeta_{\underline{d}} = \zeta_{\underline{d},n}$. We also obtain a multilinear morphism $\tilde{\zeta}_{\underline{d}} : \mathbb{W}^r \to \widetilde{W}^*$ as follows: for any $(\underline{w}_1, \dots, \underline{w}_r) \in \mathbb{W}^r$, any n, any finite subgroup G of W_n and any section a of G, we set $\tilde{\zeta}_{\underline{d},n,G}(\underline{w}_1, \dots, \underline{w}_r)(a) := \underline{E}(\zeta_{\underline{d},n}(\underline{w}_1, \dots, \underline{w}_r) \cdot \tau_n(a); 1)$, where $\underline{E}$ is the Artin-Hasse exponential and we identify G^* with $\underline{\mathrm{Hom}}(G, \mathbb{G}_m)$. It is easy to see that the composition $\mathbb{W}^r \xrightarrow{\tilde{\zeta}_{\underline{d}}} \widetilde{W}^* \twoheadrightarrow \mathbb{W}$ is equal to $\zeta_{\underline{d}}$, where $\widetilde{W}^* \twoheadrightarrow \mathbb{W}$ is the canonical projection.

Proposition 1.13. *For any finite local R-module scheme M and any $r > 1$, the following morphism is an isomorphism:*

$$\Delta_M : \bigoplus_{\underline{d} \in \mathbb{Z}_0^r} \mathrm{Hom}(\widetilde{W}^*, M) \longrightarrow \mathrm{Mult}(\mathbb{W}^r, M),$$

$$(f_{\underline{d}})_{\underline{d}} \longmapsto \sum_{\underline{d} \in \mathbb{Z}_0^r} f_{\underline{d}} \circ \tilde{\zeta}_{\underline{d}}.$$

Proof. The homomorphism Δ_M being R-linear, it is enough to show that it is a bijection and so, we can assume that $R = \mathbb{Z}$. We prove this by induction on the order of M. Assume that we have a short exact sequence $0 \to M' \to M \to M'' \to 0$ and that the result holds for M' and M''. We want to prove that it then holds for M as well. We have a commutative diagram induced by the short exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{\underline{d}} \mathrm{Hom}(\widetilde{W}^*, M') \arrow[r] \arrow[d, "\Delta_{M'}"', "\cong"] & \bigoplus_{\underline{d}} \mathrm{Hom}(\widetilde{W}^*, M) \arrow[r] \arrow[d, "\Delta_M"'] & \bigoplus_{\underline{d}} \mathrm{Hom}(\widetilde{W}^*, M'') \arrow[r] \arrow[d, "\Delta_{M''}"', "\cong"] & 0 \\
0 \arrow[r] & \mathrm{Mult}(\mathbb{W}^r, M') \arrow[r] & \mathrm{Mult}(\mathbb{W}^r, M) \arrow[r] & \mathrm{Mult}(\mathbb{W}^r, M''). &
\end{tikzcd}$$

The first row is exact because the Dieudonné module is exact and the second row is exact because Mult is left exact in each variable. The claim follows from 5-lemma. Thus, we may assume that M is a simple group scheme over k. First assume that M is local-local, and therefore isomorphic to α_p. Take a multilinear $\varphi : \mathbb{W}^r \to \alpha_p$. Since $V_{\alpha_p} = 0$, Proposition 1.8 implies that φ factors through the quotient $(\mathbb{W}/V\mathbb{W})^r$ of $\mathbb{W}^r$, yielding a multilinear morphism $\bar{\varphi} : (\varinjlim_i \alpha_{p^i})^r \cong (\mathbb{W}/V\mathbb{W})^r \to \alpha_p$. We can therefore replace $\mathbb{W}$ by $\underline{\alpha}_p := \varinjlim_i \alpha_{p^i}$ in the proposition ($\zeta_{\underline{d}}$ is now a multilinear morphism $\underline{\alpha}_p^r \to \underline{\alpha}_p$, given by

$\big((w_{i,1})_i, \dots, (w_{i,r})_i\big) \mapsto (w_{i+d_1,1} \dots w_{i+d_r,r})_i)$. Proposition A.1 implies that $\bar{\varphi}$ factors through a unique multilinear morphism $\bar{\bar{\varphi}} : \alpha_{p^n}^r \to \alpha_p$ for any $n \gg 0$, in other words, we have

$$\mathrm{Mult}(\underline{\alpha}_p^r, \alpha_p) \cong \varinjlim_n \mathrm{Mult}(\alpha_{p^n}^r, \alpha_p). \tag{1.14}$$

By Example 1.27, p. 18 of [5], we have an isomorphism $k^{n^r} \cong \mathrm{Mult}(\alpha_{p^n}^r, \mathbb{G}_a)$, given by

$$(a_{\underline{\nu}}) \mapsto \Big((x_1, \dots, x_r) \mapsto \sum_{\underline{\nu} \in [\![0, n-1]\!]^r} a_{\underline{\nu}} x_1^{p^{\nu_1}} \dots x_r^{p^{\nu_r}}\Big).$$

The image of $(a_{\underline{\nu}})$ will be a multilinear morphism into $\alpha_p \subset \mathbb{G}_a$, if and only if the p-th power vanishes, and so, if and only if $a_{\underline{\nu}} = 0$ for all $\underline{\nu}$ with $\max \underline{\nu} \neq n-1$. Renaming the indices, we deduce that multilinear morphisms $\alpha_{p^n}^r \to \alpha_p$ are unique k-linear combinations of the multilinear morphisms $(x_1, \dots, x_r) \mapsto x_1^{p^{n-1-d_1}} \dots x_r^{p^{n-1-d_r}}$ for all $\underline{d} \in \mathbb{Z}_0^r$ with $d_i < n$, which under the above isomorphism, says

$$\mathrm{Mult}(\alpha_{p^n}^r, \alpha_p) \cong \bigoplus_{\underline{d} \in \mathbb{Z}_{0,<n}^r} k. \tag{1.15}$$

We have an isomorphism $k \cong \mathrm{Hom}(\underline{\alpha}_p, \alpha_p)$, sending x to the map $(w_i)_i \mapsto x w_1$. It is now straightforward to see that the compositions

$$\bigoplus_{\underline{d} \in \mathbb{Z}_0^r} \mathrm{Hom}(\underline{\alpha}_p, \alpha_p) \cong \varinjlim_n \bigoplus_{\underline{d} \in \mathbb{Z}_{0,<n}^r} \mathrm{Hom}(\underline{\alpha}_p, \alpha_p) \cong \varinjlim_n \bigoplus_{\underline{d} \in \mathbb{Z}_{0,<n}^r} k \overset{(1.15)}{\cong} \varinjlim_n \mathrm{Mult}(\alpha_{p^n}^r, \alpha_p) \overset{(1.14)}{\cong} \mathrm{Mult}(\underline{\alpha}_p^r, \alpha_p)$$

is equal to Δ_{α_p}. Now assume that M is local-étale (and simple). By Galois descent, Δ_M is an isomorphism, if and only if it is so over $\bar{k}$. So, we may assume $k = \bar{k}$ and thus $M \cong \mu_p$. We have

$$\mathrm{Mult}(\mathbb{W}^r, \mu_p) \cong \mathrm{Mult}(\mathbb{W}^r, \mathbb{G}_m)[p] \overset{(A.1)}{\cong} \varinjlim_n \mathrm{Mult}(\mathbb{W}^{r-1} \times W_{n,n}, \mathbb{G}_m)[p] \cong \varinjlim_n \mathrm{Mult}(\mathbb{W}^{r-1}, W_{n,n})[p]$$

where we use the canonical isomorphism $W_{n,n}^* \cong W_{n,n}$. Now, applying the result for the local-local group $W_{n,n}$ (shown above), this is isomorphic to $\varinjlim_n \bigoplus_{\underline{d} \in \mathbb{Z}_0^{r-1}} \mathrm{Hom}(\mathbb{W}, W_{n,n})[p]$. This means that for every n and every multilinear morphism $\varphi : \mathbb{W}^{r-1} \times W_{n,n} \to \mathbb{G}_m$, killed by p, there are unique elements $u_{\underline{d}} \in \mathrm{Hom}(\mathbb{W}, W_{n,n})[p]$ such that for all $(\underline{w}_1, \dots, \underline{w}_r) \in \mathbb{W}^r$, we have

$$\varphi(\underline{w}_1, \dots, \underline{w}_{r-1}, \pi_n(\underline{w}_r)) = \sum_{\underline{d} \in \mathbb{Z}_0^{r-1}} \underline{E}\Big(\tilde{\zeta}_{\underline{d}}(\underline{w}_1, \dots, \underline{w}_{r-1}) \tau_n\Big(u_{\underline{d}}\big(\pi_n(\underline{w}_r)\big)\Big); 1\Big). \tag{1.16}$$

The Dieudonné module $D_*(W_{n,n})$ is isomorphic to $\mathrm{Hom}(\mathbb{W}, W_{n,n}) \cong \mathbb{E}_k/(F^n, V^n)$. Thus, the elements $u_{\underline{d}}$ (which are killed by p) can be written uniquely as $\sum_{i,j} V^{n-1-i} a_{\underline{d},i,j} F^{n-1-j}$ for $a_{i,j} \in W(k) \bmod pW(k)$ and the sum is over all i, j with $0 \leq i, j < n$ and $\min\{i, j\} = 0$. Putting this in (1.16) and using the properties of the Artin-Hasse exponential (cf. [2], Ch. III, §3), (1.16) becomes

$$\prod_{\underline{d} \in \mathbb{Z}_0^{r-1}} \prod_{0 \leq i,j < n} \underline{E}\Big(\pi_{1+i+d_1}(\underline{w}_1) \dots \pi_{1+i+d_{r-1}}(\underline{w}_{r-1}) \cdot \pi_{1+j}(\underline{w}_r) \cdot \tau_1(\tilde{a}_{\underline{d},i,j}); 1\Big)$$

with $\min\{i, j\} = 0$ and $\tilde{a}_{\underline{d},i,j}$ a Frobenius twist of $a_{\underline{d},i,j}$. The tuple $(\delta_1 \dots, \delta_r) := (i + d_1, \dots, i + d_{r-1}, j)$ belongs to $\mathbb{Z}_0^r$ with $\min\{\delta_1, \dots, \delta_{r-1}\} < n$ and $\delta_r < n$. Therefore, when n goes to infinity, every multilinear morphism $\mathbb{W}^r \to \mathbb{G}_m$ killed by p is of the form

$$(\underline{w}_1, \dots, \underline{w}_r) \mapsto \prod_{\underline{\delta} \in \mathbb{Z}_0^r} \underline{E}\big(\zeta_{\underline{\delta}}(\underline{w}_1, \dots, \underline{w}_r) \tau_1(a_{\underline{\delta}}); 1\big)$$

for unique $a_{\underline{\delta}} \in k$. Now, the isomorphism $k \cong \mathrm{Hom}(\underline{\mathbb{F}_p}, \mathbb{G}_a) \cong \mathrm{Hom}(\mathbb{G}_a^*, \mu_p)$ given by $a \mapsto (1 \mapsto \tau_1(a))$, and Cartier duality implies the proposition. $\square$

Remark 1.17. The above proposition is the adapted (covariant) version of Theorem 4.4.5 in [6] and the given proof is a modified version of its proof. Theorem 4.4.5 of [6] holds in greater generality. $\diamondsuit$

Definition 1.18. Let $M_1, ..., M_r, M, N$ be left $\mathbb{E}_k \otimes_{\mathbb{Z}} R$-modules.

(i) Denote by $\mathrm{Mult}^R(\prod_{i=1}^r M_i, N)$ the group of $W(k) \otimes_{\mathbb{Z}} R$-multilinear morphisms $\ell : \prod_{i=1}^r M_i \to N$ satisfying the following conditions for all $m_i \in M_i$:

$$\ell(Vm_1, \dots, Vm_r) = V\ell(m_1, \dots, m_r),$$

$$\forall i : \quad \ell(m_1, \dots, m_{i-1}, Fm_i, m_{i+1}, \dots m_r) = F\ell(Vm_1, \dots, Vm_{i-1}, m_i, Vm_{i+1}, \dots Vm_r).$$

(ii) Let $\mathrm{Alt}^R(M^r, N)$ denote the submodule of $\mathrm{Mult}^R(M^r, N)$ consisting of alternating morphisms. $\blacktriangle$

Remark 1.19. Let $M_1, ..., M_r, M, N$ be as above and assume that V is injective on N. Let $f : \prod_{i=1}^r M_i \to N$ be a $W(k) \otimes_{\mathbb{Z}} R$-multilinear map satisfying the V-condition above (i.e, $f(Vm_1, \dots, Vm_r) = Vf(m_1, \dots, m_r)$). Then f satisfies the F-conditions as well. Indeed, we have

$$VFf(Vx_1, \dots, Vx_{i-1}, x_i, Vx_{i+1}, \dots, Vx_r) = pf(Vx_1, \dots, Vx_{i-1}, x_i, Vx_{i+1}, \dots, Vx_r) =$$
$$f(Vx_1, \dots, Vx_{i-1}, px_i, Vx_{i+1}, \dots, Vx_r) = f(Vx_1, \dots, Vx_{i-1}, VFx_i, Vx_{i+1}, \dots, Vx_r) =$$
$$Vf(x_1, \dots, x_{i-1}, Fx_i, x_{i+1}, \dots, x_r)$$

and since by assumption V is injective on N, we can cancel V. $\diamondsuit$

Remark 1.20. For any $r > 0$ and any sheaves of R-modules M, N over k, the group $\mathrm{Mult}(\mathbb{W}^r \times M, N)$ has a multilinear left action of $\widehat{\mathbb{E}}_k^r \otimes_{\mathbb{Z}} R$ by setting $(e_1, \dots, e_r) \otimes r \cdot \varphi := \varphi \circ (e_1^* \times \cdots \times e_r^* \times r.)$, where $(\_)^*$ is the natural anti-automorphism of $\widehat{\mathbb{E}}_k$, being identity on $W(k)$ and interchanging F and V. $\diamondsuit$

Proposition 1.21. *For any $r > 0$, any finite local R-module scheme M and any finite local-local R-module schemes $M_1, \dots, M_r$, the following morphism is a well-defined isomorphism, where $D_1, \dots, D_r$ and D are respectively the covariant Dieudonné modules of $M_1, \dots, M_r$ and M:*

$$\mathrm{Mult}^R(D_1 \times \cdots \times D_r, D) \xrightarrow{\Delta_{(M_1,\dots,M_r;M)}} \mathrm{Mult}_{\widehat{\mathbb{E}}_k^r \otimes_{\mathbb{Z}} R}(D_1 \times \cdots \times D_r, \mathrm{Mult}(\mathbb{W}^r, M)),$$

$$\ell \mapsto \Delta_{(M_1,\dots,M_r;M),\ell} : (u_1, \dots, u_r) \mapsto \Delta_M\big(\ell(V^{d_1}u_1, \dots, V^{d_r}u_r)\big) = \sum_{\underline{d} \in \mathbb{Z}_0^r} \ell(V^{d_1}u_1, \dots, V^{d_r}u_r) \circ \tilde{\zeta}_{\underline{d}}.$$

Proof. Again, the proof is (almost) identical to that of Proposition 4.5.3 in [6] and we present it for completeness. All M_i being local-local, they are killed by a power of Verschiebung and so, the sum is finite. We have to show that the map is $\widehat{\mathbb{E}}_k \otimes R$-linear. Write Δ_ℓ for $\Delta_{(M_1,\dots,M_r;M),\ell}$. By symmetry, it is enough to show that $(V, 1, \dots, 1) \otimes 1 \cdot \Delta_\ell(u_1, \dots, u_r) = \Delta_\ell(Vu_1, u_2, \dots, u_r)$, $(F, 1, \dots, 1) \otimes 1 \cdot \Delta_\ell(u_1, \dots, u_r) = \Delta_\ell(Fu_1, u_2, \dots, u_r)$ and $(w, 1, \dots, 1) \otimes 1 \cdot \Delta_\ell(u_1, \dots, u_r) = \Delta_\ell(wu_1, u_2, \dots, u_r)$ for any $w \in W(k)$ and $u_i \in D_i$. We have

$$(V, 1, \dots, 1) \otimes 1 \cdot \Delta_\ell(u_1, \dots, u_r)(\underline{w}_1, \dots, \underline{w}_r) = \sum_{\underline{d} \in \mathbb{Z}_0^r} \ell(V^{d_1}u_1, \dots, V^{d_r}u_r)\big(\tilde{\zeta}_{\underline{d}}(F\underline{w}_1, \underline{w}_2, \dots, \underline{w}_r)\big)$$

$$= \sum_{\underline{d} \in \mathbb{Z}_0^r} \begin{cases} \ell(V^{d_1}u_1, \dots, V^{d_r}u_r)\big(\tilde{\zeta}_{\underline{d}-(1,0,\dots,0)}(\underline{w}_1, \underline{w}_2, \dots, \underline{w}_r)\big), & \text{if } d_1 > 0 \\ \ell(V^{d_1}u_1, \dots, V^{d_r}u_r)\big(\tilde{\zeta}_{\underline{d}+(0,1,\dots,1)}(F\underline{w}_1, F\underline{w}_2, \dots, F\underline{w}_r)\big), & \text{if } d_1 = 0 \end{cases}$$

$$= \sum_{\underline{d} \in \mathbb{Z}_0^r} \begin{cases} \ell(V^{d_1}u_1, \dots, V^{d_r}u_r)\big(\tilde{\zeta}_{\underline{d}-(1,0,\dots,0)}(\underline{w}_1, \underline{w}_2, \dots, \underline{w}_r)\big), & \text{if } d_1 > 0 \\ \ell(V^{d_1+1}u_1, \dots, V^{d_r+1}u_r)\big(\tilde{\zeta}_{\underline{d}+(0,1,\dots,1)}(\underline{w}_1, \underline{w}_2, \dots, \underline{w}_r)\big), & \text{if } d_1 = 0 \end{cases}$$

$$= \sum_{\underline{d}' \in \mathbb{Z}_0^r} \ell(V^{d'_1+1}u_1, V^{d'_2}u_2 \dots, V^{d'_r}u_r)\big(\tilde{\zeta}_{\underline{d}'}(\underline{w}_1, \underline{w}_2, \dots, \underline{w}_r)\big) = \Delta_\ell(Vu_1, u_2, \dots, u_r)(\underline{w}_1, \dots, \underline{w}_r).$$

Similar calculations show the other two equalities. It follows from Proposition 1.13 that $\Delta_{(M_1,\dots,M_r;M)}$ is injective; indeed, if $\Delta_\ell = \Delta_{\ell'}$ then, for all u_i, we have $\Delta\big(\ell(V^{d_1}u_1, \dots, V^{d_r}u_r)\big) = \Delta\big(\ell'(V^{d_1}u_1, \dots, V^{d_r}u_r)\big)$

and so, by Proposition 1.13, for all $\underline{d}$, we have $\ell(V^{d_1}u_1, \ldots, V^{d_r}u_r) = \ell'(V^{d_1}u_1, \ldots, V^{d_r}u_r)$. Setting $\underline{d} = (0, \ldots, 0)$, we get $\ell = \ell'$. It remains to show that $\Delta_{(M_1,\ldots,M_r;M)}$ is surjective. Take a multilinear morphism $f : \prod D_i \to \mathrm{Mult}(\mathbb{W}^r, M)$. By Proposition 1.13, we can write $f(u_1, \ldots, u_r) = \sum_{\underline{d}\in\mathbb{Z}_0^r} f_{\underline{d}}(u_1, \ldots, u_r) \circ \tilde{\zeta}_{\underline{d}}$ for unique $f_{\underline{d}}(u_1, \ldots, u_r) \in D$. A similar calculation as above, show that

$$f(Vu_1, u_2, \ldots, u_r) = \sum_{\underline{d}\in\mathbb{Z}_0^r} \begin{cases} f_{\underline{d}+(1,0,\ldots,0)}(u_1, \ldots, u_r) \circ \tilde{\zeta}_{\underline{d}} & \text{if } \min\{d_2, \ldots, d_r\} = 0 \\ f_{\underline{d}-(0,1,\ldots,1)}(u_1, \ldots, u_r) \circ \tilde{\zeta}_{\underline{d}} & \text{if } \min\{d_2, \ldots, d_r\} > 0. \end{cases}$$

It follows that $f_{\underline{d}}(Vu_1, u_2, \ldots, u_r) = f_{\underline{d}+(1,0,\ldots,0)}(u_1, \ldots, u_r)$ whenever $\min\{d_2, \ldots, d_r\} = 0$. By symmetry, the same relation holds in other variables and iterating this, we obtain

$$f_{\underline{d}}(u_1, u_2, \ldots, u_r) = f_{(0,\ldots,0)}(V^{d_1}u_1, \ldots, V^{d_r}u_r).$$

The same calculation shows that

$$f_{(0,\ldots,0)}(Vu_1, \ldots, Vu_r) = f_{(0,1,\ldots,1)}(Vu_1, u_2, \ldots, u_r) = Vf_{(0,\ldots,0)}(u_1, \ldots, u_r).$$

Similarly, we have $f_{(0,\ldots,0)}(Fu_1, u_2, \ldots, u_r) = Ff_{(0,\ldots,0)}(u_1, Vu_2, \ldots, Vu_r)$ and $f_{(0,\ldots,0)}(wu_1, u_2, \ldots, u_r) = wf_{(0,\ldots,0)}(u_1, \ldots, u_r)$ and their permutations (in other variables) for all $w \in W(k)$. Thus, $f_{(0,\ldots,0)} \in \mathrm{Mult}^R(D_1 \times \cdots \times D_r, D)$ and $f = \Delta_{(M_1,\ldots,M_r;M)}(f_{(0\ldots,0)})$. ☐

Proposition 1.22. *For any $r > 0$ and any finite local-local R-module schemes $M_1, \ldots, M_r$ and any sheaf of R-modules M, the following morphism is an isomorphism, where $D_1, \ldots, D_r$ are respectively the covariant Dieudonné modules of $M_1, \ldots, M_r$:*

$$\mathrm{Mult}^R(M_1 \times \cdots \times M_r, M) \xrightarrow{\nabla_{(M_1,\ldots,M_r;M)}} \mathrm{Mult}_{\widehat{\mathbb{E}}_k^r \otimes_{\mathbb{Z}} R}(D_1 \times \cdots \times D_r, \mathrm{Mult}(\mathbb{W}^r, M))$$

$$\varphi \mapsto ((f_i) \mapsto \varphi \circ (f_1 \times \cdots \times f_r)).$$

Proof. If $r = 1$, then this is Proposition A.2. For $r > 1$, the statement follows by induction and adjunction relations as in Proposition 1.7. ☐

Corollary 1.23. *For any $r > 0$, any finite local R-module scheme M and any finite local-local R-module schemes $M_1, \ldots, M_r$, there exists a natural isomorphism:*

$$\mathrm{Mult}^R(D_1 \times \cdots \times D_r, D) \longrightarrow \mathrm{Mult}^R(M_1 \times \cdots \times M_r, M),$$

where $D_1, \ldots, D_r$ and D are the covariant Dieudonné modules of $M_1, \ldots, M_r$ and M. This isomorphism is functorial in all arguments given by $\nabla^{-1}_{(M_1,\ldots,M_r;M)} \circ \Delta_{(M_1,\ldots,M_r;M)}$.

Proof. If $r = 1$, then this is the classical Dieudonné theory. If $r > 1$, M is local and M_i are local-local, then the corollary is a direct consequence of Propositions 1.22 and 1.21. ☐

Remark 1.24. 1) Last proposition and corollary are taken from [6] (Theorem 4.5.2 and Theorem 4.3.4), where they are proved for all pro-p-group schemes.

2) Let M and N be respectively a local-local and local R-module schemes. It is easy to see that the submodule $\mathrm{Alt}^R(D_*(M)^r, D_*(N))$ of $\mathrm{Mult}^R(D_*(M)^r, D_*(N))$ is mapped, under the above isomorphism, bijectively onto the submodule $\mathrm{Alt}^R(M^r, N)$ of $\mathrm{Mult}^R(M^r, N)$. ◊

2 Tensor Product and Exterior Power

In this section R is a fixed ring. We will define *tensor products* and *exterior powers* of group schemes and prove some of their properties, such as existence for finite group schemes over fields, base change, left exactness of the exterior power and relations between these constructions and covariant Dieudonné modules.

2.1 Basic constructions

Definition 2.1. Let $M_1 \cdots, M_r, M$ be R-module schemes and M' a group scheme over S.

(i) A pseudo-R-multilinear morphism $\tau : \prod_{i=1}^{r} M_i \to M'$, or by abuse of terminology, the group scheme M', is called a *tensor product of* M_i's if, for all group schemes N over S, the induced morphism

$$\tau^* : \mathrm{Hom}(M', N) \to \widetilde{\mathrm{Mult}}^R(M_1 \times \cdots \times M_r, N), \quad \psi \mapsto \psi \circ \tau,$$

is an isomorphism. If such M' and τ exist, we write $M_1 \otimes_R \cdots \otimes_R M_r$ (or $M_1 \otimes \cdots \otimes M_r$ if no confusion is likely) for M' and call τ *the universal multilinear morphism* defining $M_1 \otimes_R \cdots \otimes_R M_r$.

(ii) An alternating pseudo-R-multilinear morphism $\lambda : M^r \to M'$, or by abuse of terminology, the group scheme M', is called an r^{th} *exterior power* of M over R, if for all group schemes N over S, the induced morphism

$$\lambda^* : \mathrm{Hom}(M', N) \to \widetilde{\mathrm{Alt}}^R(M^r, N), \quad \psi \mapsto \psi \circ \lambda,$$

is an isomorphism. If such M' and λ exist, we write $\bigwedge_R^r M$ (or $\bigwedge^r M$ if no confusion is likely) for M' and call λ *the universal alternating morphism* defining $\bigwedge_R^r M$. ▲

Remark 2.2. If $M_1 \otimes \cdots \otimes M_r$ (resp. $\bigwedge^r M$) exists, it with the pseudo-R-multilinear morphism $\tau : M_1 \times \cdots \times M_r \to M_1 \otimes \cdots \otimes M_r$ (resp. $\lambda : M^r \to \bigwedge^r M$), is unique up to unique isomorphism and so, from now on, we will say "the tensor product" and "the exterior power". ◊

Definition 2.3. Let k be a field and M an R-module scheme over k. We set $M^* := \varprojlim_{M'} M'^*$, where M' runs through all finite subgroup schemes of M. ▲

Remark 2.4. Note that when M is finite over a field k, then with the above definition (notation) M^* is isomorphic to the Cartier dual of M and so there is no conflict in notations. ◊

Theorem 2.5. *Let k be a field and $M_1, \ldots, M_r$ and M be finite R-module schemes over k.*

a) $M_1 \otimes_R \cdots \otimes_R M_r$ *exists and we have* $M_1 \otimes_R \cdots \otimes_R M_r \cong \underline{\widetilde{\mathrm{Mult}}}^R(\prod_{i=1}^r M_i, \mathbb{G}_m)^*$.

b) $\bigwedge_R^r M$ *exists and we have* $\bigwedge_R^r M \cong \underline{\widetilde{\mathrm{Alt}}}^R(M^r, \mathbb{G}_m)^*$.

Proof. When $R = \mathbb{Z}$, this is Theorems 2.1.6 and 2.3.3 in [6]. The proof of the general case is not different and therefore we only sketch the proof (as given in [6]). Let N be a finite group scheme over k. Then, using Proposition A.1, we have functorial isomorphisms

$$\mathrm{Hom}(\underline{\widetilde{\mathrm{Mult}}}^R(\prod_i M_i, \mathbb{G}_m)^*, N) \overset{(A.1)}{\cong} \varinjlim_{M'} \mathrm{Hom}(M'^*, N) \cong \varinjlim_{M'^*} \mathrm{Hom}(N^*, M') \overset{(A.1)}{\cong}$$

$$\mathrm{Hom}(N^*, \underline{\widetilde{\mathrm{Mult}}}^R(\prod_i M_i, \mathbb{G}_m)) \overset{(1.7)}{\cong} \widetilde{\mathrm{Mult}}^R(\prod_i M_i, \underline{\mathrm{Hom}}(N^*, \mathbb{G}_m)) \cong \widetilde{\mathrm{Mult}}^R(\prod_i M_i, N).$$

By Proposition A.1, the same holds for N affine of finite type. By dévissage, we reduce the general case to the case of affine group schemes of finite type (cf. [9], §3.3, p.24). Since the above isomorphisms are functorial in N, we obtain a universal multilinear morphism $\tau : \prod_i M_i \to \underline{\widetilde{\mathrm{Mult}}}^R(\prod_i M_i, \mathbb{G}_m)^*$. This proves a). The proof of b) is similar. □

Remark 2.6. Let $\varphi_i : M_i \to N_i$ for $i = 1, \cdots, r$ and $\varphi : M \to N$ be R-linear morphisms.

1) There exists a unique homomorphism $\varphi_1 \otimes \cdots \otimes \varphi_r : M_1 \otimes \cdots \otimes M_r \to N_1 \otimes \cdots \otimes N_r$ such that the following diagram commutes:

$$\begin{tikzcd}
M_1 \times \cdots \times M_r \arrow[r, "\varphi_1 \times \cdots \times \varphi_r"] \arrow[d, "\tau"'] & N_1 \times \cdots \times N_r \arrow[d, "\tau'"] \\
M_1 \otimes \cdots \otimes M_r \arrow[r, dashed, "\varphi_1 \otimes \cdots \otimes \varphi_r"'] & N_1 \otimes \cdots \otimes N_r.
\end{tikzcd}$$

2) Similarly, for all r, we obtain a homomorphism $\bigwedge^r \varphi : \bigwedge^r M \to \bigwedge^r N$. $\Diamond$

Remark 2.7. Let $M_1, \cdots, M_r$ and M be R-module schemes and $x \in R$.

1) From the definition of the tensor product, it follows that if $M_1 \otimes \cdots \otimes M_n$ exists, then it possesses an R-module structure, where x acts via the morphism

$$\mathrm{Id}_{M_1} \otimes \cdots \otimes \mathrm{Id}_{M_{i-1}} \otimes (x.) \otimes \mathrm{Id}_{M_{i+1}} \otimes \cdots \otimes \mathrm{Id}_{M_r} \qquad \text{any} i.$$

2) Similarly, if $\bigwedge^r M$ exists, then the action of x is given by the following commutative diagram:

$$\begin{tikzcd} M \times \cdots \times M \arrow[r, "\mathrm{Id}_M \times \cdots \times (x.) \times \cdots \times \mathrm{Id}_M"] \arrow[d, "\lambda"'] & M \times \cdots \times M . \arrow[d, "\lambda"] \\ \bigwedge^r M \arrow[r, "r."'] & \bigwedge^r M \end{tikzcd}$$

$\Diamond$

Remark 2.8. It follows from Remark 2.7, that if the tensor product $M_1 \otimes \cdots \otimes M_r$ exists, then it satisfies the following universal property as well: let N be an R-module scheme, then the morphism

$$\mathrm{Hom}^R(M_1 \otimes \cdots \otimes M_r, N) \to \mathrm{Mult}^R(M_1 \times \cdots \times M_r, N), \quad \varphi \mapsto \varphi \circ \tau$$

is an isomorphism. In fact we have the following commutative diagram:

$$\begin{tikzcd} \mathrm{Hom}^R(M_1 \otimes \cdots \otimes M_r, N) \arrow[r, "\cong"] \arrow[d, hook] & \mathrm{Mult}^R(M_1 \times \cdots \times M_r, N) \arrow[d, hook] \\ \mathrm{Hom}(M_1 \otimes \cdots \otimes M_r, N) \arrow[r, "\cong"'] & \widetilde{\mathrm{Mult}}^R(M_1 \times \cdots \times M_r, N). \end{tikzcd}$$

$\Diamond$

Proposition 2.9. *Let $R \twoheadrightarrow R'$ be a surjective ring homomorphism, $M_1, \ldots, M_r$ and M be R'-module schemes over S. Then we have canonical R-linear isomorphisms:*

a) $M_1 \otimes_R \cdots \otimes_R M_r \cong M_1 \otimes_{R'} \cdots \otimes_{R'} M_r$.

b) $\bigwedge\limits_R^n M \cong \bigwedge\limits_{R'}^n M$.

Proof. We show the first isomorphism; the second one can be shown similarly. We prove that the tensor product $\tau' : \prod_{i=1}^r M_i \to M_1 \otimes_{R'} \cdots \otimes_{R'} M_r$ has the universal property of the tensor product of M_i over R. So, take an R-module scheme N and an R-multilinear morphism $\varphi : \prod_{i=1}^r M_i \to N$. Let I be the kernel of the homomorphism $R \to R'$ and set $K := \bigcap_{r \in I} \mathrm{Ker}(r. : N \to N)$. It is clear from the definition of K that it is an R'-module scheme. Since the R-module structure of M_i is given by the "restriction of scalars" $R \to R'$, and so $I \cdot M_i = 0$, the R-multilinear morphism φ factors uniquely through $\iota : K \hookrightarrow N$:

$$\begin{tikzcd} & M_1 \times \cdots \times M_r \arrow[dl, dashed, "\overline{\varphi}"'] \arrow[dr, "\varphi"] & \\ K \arrow[rr, hook, "\iota"'] & & N. \end{tikzcd}$$

The morphism $\overline{\varphi}$ is R'-multilinear. So, there is a unique R'-linear morphism $\varphi' : M_1 \otimes_{R'} \cdots \otimes_{R'} M_r \to K$ such that $\varphi' \circ \tau' = \overline{\varphi}$. Putting this with the last diagram, we obtain:

$$\begin{tikzcd} & M_1 \times \cdots \times M_r \arrow[dl, "\tau'"'] \arrow[d, "\overline{\varphi}"] \arrow[dr, "\varphi"] & \\ M_1 \otimes_{R'} \cdots \otimes_{R'} M_r \arrow[r, "\varphi'"'] & K \arrow[r, hook, "\iota"'] & N. \end{tikzcd}$$

So, the composition $\iota \circ \varphi'$ is R-linear and its composition with τ' is equal to the given multilinear morphism φ. This shows the existence part of the universal property of the tensor product. The uniqueness part follows from the uniqueness of φ' and $\overline{\varphi}$ and the fact that the homomorphism $\iota : K \to N$ is a monomorphism (note that since $M_1 \otimes_{R'} \cdots \otimes_{R'} M_r$ is an R'-module scheme, every R-homomorphism from $M_1 \otimes_{R'} \cdots \otimes_{R'} M_r$ to N factors uniquely through $\iota : K \hookrightarrow N$). □

2.2 Base change

Definition 2.10. Let M and $M_1, \cdots, M_r$ be R-module schemes over a base scheme S and let T be an S-scheme and r a positive natural number. Then we have a natural isomorphism $\prod_{i=1}^r M_{i,T} \cong (\prod_{i=1}^r M_i)_T$.

(i) The universal multilinear morphisms defining $M_1 \otimes \cdots \otimes M_r$ and $M_{1_T} \otimes \cdots \otimes M_{r_T}$ give rise to the following diagram

$$\begin{tikzcd} M_{1,T} \times_T \cdots \times_T M_{r,T} \arrow[r, "\cong"] \arrow[d, "\tau"'] & (M_1 \times_S \cdots \times_S M_r)_T \arrow[d, "\tau_T"] \\ M_{1,T} \otimes \cdots \otimes M_{r,T} \arrow[r, dashed, "\tau_{T/S}"'] & (M_1 \otimes \cdots \otimes M_r)_T. \end{tikzcd}$$

The universal property of $M_{1,T} \otimes \cdots \otimes M_{r,T}$ fills the diagram by a unique morphism $\tau_{T/S} : M_{1,T} \otimes \cdots \otimes M_{r,T} \to (M_1 \otimes \cdots \otimes M_r)_T$ called *the base change homomorphism of tensor product.*

(ii) Similarly, we get *the base change homomorphism of exterior power* $\lambda_{T/S} : \bigwedge^n(M_T) \to (\bigwedge^n M)_T$. ▲

Remark 2.11. The base change homomorphisms in the last definition need not be isomorphisms. However, as we will show (cf. Propositions 2.12 and 2.15), if $S = \mathbf{Spec}\, E$ and $T = \mathbf{Spec}\, L$, where L/E is either a separable or a finite field extension, then the base change homomorphisms are isomorphisms. ◊

Proposition 2.12. *Let E be a field and L/E a separable field extension. Then the base change homomorphisms $M_{1,L} \otimes \cdots \otimes M_{r,L} \to (M_1 \otimes \cdots \otimes M_r)_L$ and $\bigwedge^r(M_L) \to (\bigwedge^r M)_L$ are isomorphisms.*

Proof. Set $G := \widetilde{\underline{\mathrm{Mult}}}^R(M_1 \times \cdots \times M_r, \mathbb{G}_m)$. In view of Theorem 2.5, it is sufficient to prove that the system of finite subgroups of G_L, which are of the form H_L for a finite subgroup H of G, is cofinal in the system of all finite subgroups of G_L. This follows from a standard Galois descent argument. □

Note that it follows from the adjunction of the Weil restriction that if N is an affine R-module scheme over L, then $\mathrm{Res}_{L/E} N$ is an affine R-module scheme over E and if M is an R-module scheme over E, then there is a canonical R-linear isomorphism $\mathrm{Hom}_L^R(M_L, N) \cong \mathrm{Hom}_E^R(M, \mathrm{Res}_{L/E} N)$.

Lemma 2.13. *Let E be a field and L/E a finite field extension. Let M be an affine R-module scheme over E and N an fppf sheaf of R-modules over* $\mathbf{Spec}\, L$*. Then there exists a canonical sheaf isomorphism*

$$\mathrm{Res}_{L/E} \underline{\mathrm{Hom}}_L(M_L, N) \cong \underline{\mathrm{Hom}}_E(M, \mathrm{Res}_{L/E} N)$$

which is the "sheafified" version of the Weil restriction.

Proof. This follows from the definition of the Weil restriction and its properties. □

Proposition 2.14. *Let L/E a finite extension of fields. Let $M_1, M_2, \ldots, M_r$ and M be affine R-module schemes over E and N an fppf sheaf of Abelian groups over L.*

a) The bijection $\mathrm{Mor}_L(\prod_{i=1}^r M_{i,L}, N) \cong \mathrm{Mor}_E(\prod_{i=1}^r M_i, \mathrm{Res}_{L/E} N)$ *restricts to an isomorphism*

$$\widetilde{\mathrm{Mult}}_L^R(\prod_{i=1}^r M_{i,L}, N) \cong \widetilde{\mathrm{Mult}}_E^R(\prod_{i=1}^r M_i, \mathrm{Res}_{L/E} N).$$

b) The isomorphism $\widetilde{\mathrm{Mult}}_L^R(M_L^r, N) \cong \widetilde{\mathrm{Mult}}_E^R(M^r, \mathrm{Res}_{L/E} N)$ *of part a) restricts to an isomorphism*

$$\widetilde{\mathrm{Alt}}_L^R(M_L^r, N) \cong \widetilde{\mathrm{Alt}}_E^R(M, \mathrm{Res}_{L/E} N).$$

Proof. a) This follows from previous lemma and induction on r.

b) By the adjunction property, we know that there exists a unique R-linear homomorphism $\sigma_N : (\operatorname{Res}_{L/E} N)_L \to N$ with the following universal property: for every E-scheme X the map

$$\operatorname{Mor}_E(X, \operatorname{Res}_{L/E} N) \to \operatorname{Mor}_L(X_L, N)$$

induced by base change to L and composing with σ_N is an isomorphism. It follows that if $\varphi : M^r \to \operatorname{Res}_{L/E} N$ is alternating, the morphism $\sigma_N \circ \varphi_L$ is alternating too and therefore, the R-linear isomorphism of part a) restricts to an R-linear monomorphism

$$\widetilde{\operatorname{Alt}}^R_E(M^r, \operatorname{Res}_{L/E} N) \hookrightarrow \widetilde{\operatorname{Alt}}^R_L(M^r_L, N).$$

We have to show that this is surjective too. Take an alternating morphism $\varphi : M^r_L \to N$. Because of part a), we know that there exists a pseudo-R-multilinear morphism $\psi : M^r \to \operatorname{Res}_{L/EN}$ such that $\varphi = \sigma_N \circ \psi_L$. We want to show that ψ is alternating. For any $1 \le i < j \le r$, let

$$\Delta^r_{i,j} : M^{r-1} \to M^r, \quad (m_1, \ldots, m_{r-1}) \mapsto (m_1, \ldots, m_{j-1}, m_i, m_j, \ldots, m_r)$$

denote the generalized diagonal embedding equating the i^{th} and j^{th} components. By functoriality of the Weil restriction, the following diagram commutes:

$$\begin{tikzcd}
\widetilde{\operatorname{Mult}}^R_E(M^r, \operatorname{Res}_{L/E} N) \arrow[r, "\cong"] \arrow[d, "\Delta^*_{i,j}"'] & \widetilde{\operatorname{Mult}}^R_L(M^r_L, N) \arrow[d, "\Delta^*_{i,j}"] \\
\widetilde{\operatorname{Mult}}^R_E(M^{r-1}, \operatorname{Res}_{L/E} N) \arrow[r, "\cong"'] & \widetilde{\operatorname{Mult}}^R_L(M^{r-1}_L, N).
\end{tikzcd}$$

Since φ is mapped to zero under the homomorphism $\widetilde{\operatorname{Mult}}^R_L(M^r_L, N) \to \widetilde{\operatorname{Mult}}^R_L(M^{r-1}_L, N)$ (because it is alternating), the morphism ψ lies in the kernel of the homomorphism

$$\widetilde{\operatorname{Mult}}^R_E(M^r, \operatorname{Res}_{L/E} N) \to \widetilde{\operatorname{Mult}}^R_E(M^{r-1}, \operatorname{Res}_{L/E} N),$$

and this holds for every pair $i < j$. Hence ψ is alternating.

□

Proposition 2.15. *Let L/E a finite extension of fields. Let $M_1, M_2, \ldots, M_r$ and M be finite R-module schemes over E. Then the base change homomorphisms $M_{1,L} \otimes \cdots \otimes M_{r,L} \to (M_1 \otimes \cdots \otimes M_r)_L$ and $\bigwedge^r(M_L) \to (\bigwedge^r M)_L$ are isomorphisms.*

Proof. We prove the statement for the tensor product and drop the proofs for the exterior power, as it can be proved similarly. Let N be an affine group scheme over L. By functoriality of the Weil restriction, we have the following commutative diagram:

$$\begin{tikzcd}
\operatorname{Hom}_E(M_1 \otimes \cdots \otimes M_r, \operatorname{Res}_{L/E} N) \arrow[r, "\cong"] \arrow[d, "\cong"'] & \operatorname{Hom}_L((M_1 \otimes \cdots \otimes M_r)_L, N) \arrow[d] \\
\widetilde{\operatorname{Mult}}^R_E(M_1 \times_R \cdots \times_R M_r, \operatorname{Res}_{L/E} N) \arrow[r, "\cong"'] & \widetilde{\operatorname{Mult}}^R_L(M_{1,L} \times_R \cdots \times_R M_{r,L}, N)
\end{tikzcd}$$

where the vertical homomorphisms are induced by the universal multilinear morphisms. The horizontal morphisms are isomorphisms by previous proposition. Thus the right vertical homomorphism is an isomorphism as well, which implies that the base change homomorphism is an isomorphism. □

2.3 Main properties of exterior powers

In this section all schemes are over a base scheme S.

Proposition 2.16. *Let M be an R-module scheme. If $\bigwedge^r M = 0$ then, $\forall s \geq r$, $\bigwedge^s M = 0$.*

Proof. We show that $\bigwedge^{r+1} M = 0$; the result follows by induction. For all N, we have

$$\mathrm{Hom}^R(\bigwedge^{r+1} M, N) \cong \mathrm{Alt}^R(M^{r+1}, N) \cong \mathrm{Alt}^R(M^r, \underline{\mathrm{Hom}}^R(M, N)) = \mathrm{Hom}^R(\bigwedge^r M, \underline{\mathrm{Hom}}^R(M, N)) = 0.$$

□

Proposition 2.17. *Let M_1, M_2 and P be R-module schemes. We have a natural isomorphism*

$$\lambda_{1,2}^* : \mathrm{Alt}^R(M_1^{r_1} \times M_2^{r_2}, P) \cong \mathrm{Hom}^R(\bigwedge^{r_1} M_1 \otimes \bigwedge^{r_2} M_2, P).$$

Proof. Standard. □

Lemma 2.18. *Let $\varphi_i : M_i \to M,\ i = 1, 2$ be epimorphisms of R-module schemes. Then the morphism*

$$\bigwedge^{r_1} \varphi_1 \wedge \bigwedge^{r_2} \varphi_2 : \bigwedge^{r_1} M_1 \otimes \bigwedge^{r_2} M_2 \to \bigwedge^{r_1+r_2} M$$

is an epimorphism. In particular, we have a canonical epimorphism

$$\bigwedge^{r_1} M \otimes \bigwedge^{r_2} M \twoheadrightarrow \bigwedge^{r_1+r_2} M.$$

Proof. The morphism $\varphi_1^{r_1} \times \varphi_2^{r_2} : M_1^{r_2} \times M_2^{r_2} \to M^{r_1+r_2}$ is an epimorphism. Therefore, for every R-module scheme P, the induced morphism $\mathrm{Alt}^R(M^{r_1+r_2}, P) \to \mathrm{Alt}^R(M_1^{r_2} \times M_2^{r_2}, P)$ is injective. The lemma now follows from the previous proposition and the universal property of alternating powers. □

Notations 2.19. Let M', M'' be sub-R-module schemes of M and P and N R-module schemes. By $\mathrm{Alt}^R(M'^r \times M''^s \times P^t, N)$ we mean the module of R-multilinear morphisms that are alternating in M'^r, M''^s and $(M' \cap M'')^{r+s}$ (as a submodule scheme of M^{r+s})) and in P^t. When we say that a multilinear morphism $M'^r \times M''^s \times P^t \to N$ is alternating, we mean that it belongs to the module $\mathrm{Alt}^R(M'^r \times M''^s \times P^t, N)$. Likewise, we define the module $\mathrm{Alt}^R(M_1^{r_1} \times \cdots \times M_n^{r_n} \times P_1^{s_1} \times \cdots \times P_m^{s_m}, N)$ with M_i sub-R-module schemes of M and P_j's arbitrary R-module schemes.

Lemma 2.20. *Let $\pi : M \twoheadrightarrow M''$ be an epimorphism and let $\varphi : M''^r \to H$ be an R-multilinear morphism such that the composition $\varphi \circ \pi^r : M^r \to H$ is alternating. Then φ is alternating as well.*

Proof. Standard. □

Remark 2.21. Let M' be a sub-R-module scheme of M and $\pi : M \twoheadrightarrow M''$ an epimorphism. It can be shown in the same fashion that if the composition of a multilinear morphism $M''^r \times M^s \times M'^t \to N$ with the epimorphism $\pi^r \times \mathrm{Id}_{M'^s} \times \mathrm{Id}_{M'^t} : M^r \times M^s \times M'^t \to M''^r \times M^s \times M'^t$ is alternating (in the sense of the Notations 2.19), then this multilinear morphism is also alternating. ◊

Lemma 2.22. *Let $M_1 \cdots, M_r$ be R-module schemes and $\psi : \prod_{i=1}^r M_i \to N$ an R-multilinear morphism. Assume that for some $1 \leq i \leq r$ we have an exact sequence $M_i' \xrightarrow{\iota} M_i \xrightarrow{\pi} M_i'' \to 0$. If the restriction $\psi|_{M_1 \times \cdots \times M_i' \times \cdots \times M_r}$ is zero, then there is a unique multilinear morphism $\psi' : M_1 \times \cdots \times M_i'' \times \cdots \times M_r \to N$ such that $\psi = \psi' \circ (\mathrm{Id}_{M_1} \times \cdots \times \pi \times \cdots \times \mathrm{Id}_{M_r})$ with π at the i^{th} place.*

Proof. This follows from the left exactness of the Hom functor. □

Lemma 2.23. *Let $M_1 \xrightarrow{\iota} M_2 \xrightarrow{\pi} M_3 \to 0$ be an exact sequence of R-module schemes. Then the sequence $0 \to \mathrm{Alt}^R(M_3^r, N) \to \mathrm{Alt}^R(M_2^r, N) \to \mathrm{Alt}^R(M_1 \times M_2^{r-1}, N)$ is exact.*

Proof. This follows from Lemmas 2.20 and 2.22 □

Lemma 2.24. *Let $M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3 \to 0$ be an exact sequence of R-module schemes. Then the sequence $M_1 \otimes \bigwedge^{r-1} M_2 \xrightarrow{\alpha \wedge \bigwedge^{r-1} \mathrm{Id}_{M_2}} \bigwedge^r M_2 \xrightarrow{\bigwedge^r \beta} \bigwedge^r M_3 \to 0$ is exact.*

Proof. This follows from Lemma 2.23 and Proposition 2.17. ☐

Proposition 2.25. *Let M be an R-module scheme, and Q the cokernel of multiplication by an element $x \in R$, i.e., we have an exact sequence $M \xrightarrow{x.} M \xrightarrow{p} Q \to 0$. Then the following sequence is exact:*

$$\bigwedge^r M \xrightarrow{x.} \bigwedge^r M \xrightarrow{\bigwedge^r p} \bigwedge^r Q \to 0.$$

Proof. We have a commutative diagram

$$\begin{tikzcd} M \otimes \bigwedge^{r-1} M \arrow[rr, "(x.)\wedge\bigwedge^{r-1}\mathrm{Id}"] \arrow[dr, "\mathrm{Id}\wedge\bigwedge^{r-1}\mathrm{Id}"'] & & \bigwedge^r M \\ & \bigwedge^r M. \arrow[ur, "x."'] & \end{tikzcd} \tag{2.26}$$

Now, apply Lemma 2.24 to the exact sequence $M \xrightarrow{x.} M \xrightarrow{p} Q \to 0$ to obtain the exact sequence:

$$M \otimes \bigwedge^{r-1} M \xrightarrow{(x.)\wedge\bigwedge^{r-1}\mathrm{Id}_M} \bigwedge^r M \xrightarrow{\bigwedge^r p} \bigwedge^r Q \to 0.$$

Using diagram 2.26, we can factorize the first morphism of the sequence, so that the following diagram is commutative with exact row

$$\begin{tikzcd} M \otimes \bigwedge^{r-1} M \arrow[rr, "(x.)\wedge\bigwedge^{r-1}\mathrm{Id}"] \arrow[dr, "\mathrm{Id}\wedge\bigwedge^{r-1}\mathrm{Id}"'] & & \bigwedge^r M \arrow[r, "\bigwedge^r p"] & \bigwedge^r Q \arrow[r] & 0 \\ & \bigwedge^r M. \arrow[ur, "x."'] & & & \end{tikzcd}$$

Since the morphism $\mathrm{Id}\wedge\bigwedge^{r-1}\mathrm{Id} : M\otimes\bigwedge^{r-1} M \to \bigwedge^r M$ is an epimorphism, we conclude that the sequence $\bigwedge^r M \xrightarrow{x.} \bigwedge^r M \xrightarrow{\bigwedge^r p} \bigwedge^r Q \to 0$ is exact as well, and the proof is achieved. ☐

2.4 Dieudonné modules

In this subsection R is a ring and k is a perfect field of characteristic $p > 2$.

Definition 2.27. Let P be an R-module endowed with four set-theoretic maps $F, V : P \to P$ and $\varphi, \upsilon : \bigwedge_R^r P \to \bigwedge_R^r P$. Denote by λ the alternating morphism $\lambda : P \times \cdots \times P \to \bigwedge_R^r P$ which sends $(x_1, \cdots, x_r)$ to $x_1 \wedge \cdots \wedge x_r$.

(i) The following diagram is called the *F-diagram associated to φ*

$$\begin{tikzcd} P \times P \times \cdots \times P \arrow[r, "\lambda"] & \bigwedge_R^r P \arrow[dd, "\varphi"] \\ P \times P \times \cdots \times P \arrow[u, "\mathrm{Id}\times V\times\cdots\times V"] \arrow[d, "F\times\mathrm{Id}\times\cdots\times\mathrm{Id}"'] & \\ P \times P \times \cdots \times P \arrow[r, "\lambda"'] & \bigwedge_R^r P. \end{tikzcd}$$

(ii) The following diagram is called the *V-diagram associated to υ*

$$\begin{tikzcd} P \times P \times \cdots \times P \arrow[r, "\lambda"] \arrow[d, "V\times\cdots\times V"'] & \bigwedge_R^r P \arrow[d, "\upsilon"] \\ P \times P \times \cdots \times P \arrow[r, "\lambda"'] & \bigwedge_R^r P. \end{tikzcd}$$

▲

Definition 2.28. Let $M_1, \cdots, M_r$ and M be left $\mathbb{E}_k \otimes_{\mathbb{Z}} R$-modules.

(i) Consider the tensor product $\mathbb{E}_k \otimes_W M_1 \otimes_{W\otimes_{\mathbb{Z}} R} \cdots \otimes_{W\otimes_{\mathbb{Z}} R} M_r$ which uses the action W on $\mathbb{E}_k$ by right multiplication. This is a left $\mathbb{E}_k \otimes_{\mathbb{Z}} R$-module with respect to left multiplication of $\mathbb{E}_k$ on the first factor and the action of R on the other factors. Define $T(M_1 \times \cdots \times M_r)$ to be its quotient by the $\mathbb{E}_k$-submodule generated by the elements (for all $m_i \in M_i$)

$$V \otimes m_1 \otimes \cdots \otimes m_r - 1 \otimes Vm_1 \otimes \cdots \otimes Vm_r,$$

$$\forall i: \quad F\otimes Vm_1\otimes\cdots\otimes Vm_{i-1}\otimes m_i\otimes Vm_{i+1}\otimes\cdots\otimes Vm_r - 1\otimes m_1\otimes\cdots\otimes m_{i-1}\otimes Fm_i\otimes m_{i+1}\otimes\cdots\otimes m_r$$

(ii) Define $T_{\mathrm{alt}}(M^r)$ to be the quotient of $T(M^r)$ by the $\mathbb{E}_k \otimes_{\mathbb{Z}} R$-submodule generated by the elements $[1 \otimes m_1 \otimes \cdots \otimes m_r]$ for all $m_i \in M$.

Remark 2.29. Let $M_1, \ldots, M_r, M$ be $\mathbb{E}_k \otimes_{\mathbb{Z}} R$-modules which are of finite length as a $W \otimes_{\mathbb{Z}} R$-module.

1) There is a canonical morphism $\tau : \prod_{i=1}^r M_i \to T(M_1 \times \cdots \times M_r), (m_1, \ldots, m_r) \mapsto [1 \otimes m_1 \otimes \cdots \otimes m_r]$. This morphism belongs to the R-module $\mathrm{Mult}^R(M_1 \times \cdots \times M_r, T(M_1 \times \cdots \times M_r))$ and has the universal property that for any $\mathbb{E}_k \otimes_{\mathbb{Z}} R$-module N, τ induces an isomorphism

$$\tau^* : \mathrm{Hom}(T(M_1 \times \cdots \times M_r), N) \to \mathrm{Mult}^R(M_1 \times \cdots \times M_r, N).$$

2) Similarly, the canonical morphism $\lambda : M^r \to T_{\mathrm{alt}}(M^r)$, $(m_1, \ldots, m_r) \mapsto [1 \otimes m_1 \otimes \cdots \otimes m_r]$ induces an isomorphism $\lambda^* : \mathrm{Hom}(T_{\mathrm{alt}}(M^r), N) \to \mathrm{Alt}^R(M^r, N)$. $\lozenge$

Lemma 2.30. *Let P be an $\mathbb{E}_k \otimes_{\mathbb{Z}} R$-module which is of finite length as a $W \otimes_{\mathbb{Z}} R$-module. Denote $\bigwedge^r_{W\otimes_{\mathbb{Z}} R} P$ by $\bigwedge^r P$. Assume that we have two commuting maps $\varphi : \bigwedge^r P \to \bigwedge^r P$ and respectively $\upsilon : \bigwedge^r P \to \bigwedge^r P$ which are ${}^{\sigma^{-1}} \otimes \mathrm{Id}$-linear and respectively ${}^{\sigma} \otimes \mathrm{Id}$-linear and make the F-diagram associated to φ and respectively the V-diagram associated to υ commute. Assume further that every element of $\bigwedge^r P$ is a linear combination of elements of the form $d_1 \wedge Vd_2 \wedge \cdots \wedge Vd_r$. Then there is a natural structure of $\mathbb{E}_k$-module on $\bigwedge^r P$, where F and V act through φ and υ and we have a canonical $\mathbb{E}_k \otimes_{\mathbb{Z}} R$-linear isomorphism $T_{alt}(P^r) \cong \bigwedge^r P$.*

Proof. We first show that $\varphi \circ \upsilon = p$. Indeed, we have for all $d_1, \cdots, d_r \in P$

$$\varphi \circ \upsilon(d_1 \wedge \cdots \wedge d_r) = \varphi(Vd_1 \wedge \cdots \wedge Vd_r) = FVd_1 \wedge d_2 \wedge \cdots \wedge d_r = pd_1 \wedge d_2 \wedge \cdots \wedge d_r$$

where the first equality follows from the V-diagram and the second equality from the F-diagram. As $\upsilon \circ \varphi = \varphi \circ \upsilon$, we have $\upsilon \circ \varphi = \varphi \circ \upsilon = p$. By definition, $T_{\mathrm{alt}}(P^r)$ is the quotient of $\mathbb{E}_k \otimes_W \bigwedge^r P$ by the submodule generated by the relations:

$$(\rho_1) \qquad V \otimes m_1 \wedge \cdots \wedge m_r - 1 \otimes Vm_1 \wedge \cdots \wedge Vm_r,$$

$$(\rho_2) \qquad F \otimes m_1 \wedge Vm_2 \wedge \cdots \wedge Vm_r - 1 \otimes Fm_1 \wedge m_2 \wedge \cdots \wedge m_r$$

(note that the other relations follow from these two). Define a morphism

$$\theta : \mathbb{E}_k \otimes_W \bigwedge^r P \to \bigwedge^r P, \qquad F^i \otimes x \mapsto \varphi^i(x) \quad \text{and} \quad V^i \otimes x \mapsto \upsilon^i(x).$$

Since $\upsilon\circ\varphi = \varphi\circ\upsilon = p$, this morphism is a well defined $\mathbb{E}_k$-linear morphism. We claim that this morphism factors through the quotient $T_{\mathrm{alt}}(P^r)$. i.e., it is zero on the relations ρ_1 and ρ_2.

(ρ_1) : $\theta(V \otimes m_1 \wedge \cdots \wedge m_r - 1 \otimes Vm_1 \wedge \cdots \wedge Vm_r) = \upsilon(m_1 \wedge \cdots \wedge m_r) - Vm_1 \wedge \cdots \wedge Vm_r = 0$ by the V-diagram.

(ρ_2) : $\theta(F \otimes m_1 \wedge Vm_2 \wedge \cdots \wedge Vm_r - 1 \otimes Fm_1 \wedge m_2 \wedge \cdots \wedge m_r) =$
$\varphi(m_1 \wedge Vm_2 \wedge \cdots \wedge Vm_r) - Fm_1 \wedge m_2 \wedge \cdots \wedge m_r = 0$ by the F-diagram.

It is straightforward to see that the morphism $\vartheta : \bigwedge^r P \to T_{\mathrm{alt}}(P^r)$ sending an element x to $[1 \otimes x]$ is an inverse of $\bar{\theta} : T_{\mathrm{alt}}(P^r) \to \bigwedge^r P$ induced by θ. Therefore we have $T_{\mathrm{alt}}(P^r) \cong \bigwedge^r P$. $\square$

Lemma 2.31. *Let P, Q be two $\mathbb{E}_k \otimes_{\mathbb{Z}} R$-modules which are of finite length as $W \otimes_{\mathbb{Z}} R$-modules. Assume further that the multiplication by V on P is an isomorphism. Then there exists a natural structure of $\mathbb{E}_k$-module on $P \otimes_{W \otimes_{\mathbb{Z}} R} Q$, where F acts as $V^{-1} \otimes F$ and V acts as $V \otimes V$. Furthermore, we have a canonical $\mathbb{E}_k \otimes_{\mathbb{Z}} R$-linear isomorphism $T(P \times Q) \cong P \otimes_{W \otimes_{\mathbb{Z}} R} Q$.*

Proof. The compositions $(V^{-1} \otimes F) \circ (V \otimes V)$ and $(V \otimes V) \circ (V^{-1} \otimes F)$ are equal to multiplication by p and so there is a natural structure of $\mathbb{E}_k$-module on the tensor product $P \otimes_{W \otimes_{\mathbb{Z}} R} Q$. It remains to show the isomorphism. Define morphisms $\theta : T(P \times Q) \to P \otimes_{W \otimes_{\mathbb{Z}} R} Q$ and $\eta : P \otimes_{W \otimes_{\mathbb{Z}} R} Q \to T(P \times Q)$ as follows: $\theta([1 \otimes x \otimes y]) = x \otimes y, \theta([F^i \otimes x \otimes y]) = V^{-i}(x) \otimes F^i(y)$ and $\theta([V^i \otimes x \otimes y]) = V^i(x) \otimes V^i(y)$. For η, set $\eta(x \otimes y) = [1 \otimes x \otimes y]$, where by elements in brackets, we mean their class in the quotient $T(P \times Q)$. It is straightforward to check that θ is well-defined, and these morphisms are inverse to each other. $\square$

Proposition 2.32. *Let G_1, G_2 and G be finite p-torsion R-module schemes over k, then the tensor product $G_1 \otimes G_2$ and the exterior power $\bigwedge^r G$ exist and are again finite p-torsion R-module schemes over k, and there are natural isomorphisms*

a) $D_*(G_1 \otimes G_2) \cong T(D_*(G_1) \times D_*(G_2))$,

b) $D_*(\bigwedge^r G) \cong T_{alt}(D_*(G)^r)$.

Proof. The existence is given by Theorem 2.5. The isomorphism in *a*) is the formal consequence of the functorial (in H) isomorphisms

$$\mathrm{Hom}_{W\otimes_{\mathbb{Z}}R}(D_*(G_1 \otimes G_2), D_*(H)) \cong \mathrm{Hom}^R(G_1 \otimes G_2, H) \cong \mathrm{Mult}^R(G_1 \times G_2, H) \overset{(1.23)}{\cong}$$

$$\mathrm{Mult}^R(D_*(G_1) \times D_*(G_2), D_*(H)) \overset{(2.29)}{\cong} \mathrm{Hom}_{W\otimes_{\mathbb{Z}}R}(T(D_*(G_1) \times D_*(G_2)), D_*(H)).$$

The proof of *b*) is similar. $\square$

Remark 2.33. Let M be a finite p-torsion R-module scheme over k. According to Remark 1.24, we have an isomorphism

$$\mathrm{Alt}^R(D_*(M)^r, D_*(\bigwedge^r M)) \cong \mathrm{Alt}^R(M^r, \bigwedge^r M). \tag{2.34}$$

By Remark 2.29 the universal morphism $\lambda : D_*(M)^r \to T_{\mathrm{alt}}(D_*(M)^r)$ induces an isomorphism

$$\mathrm{Hom}^R(T_{\mathrm{alt}}(D_*(M)^r), D_*(\bigwedge^r M)) \to \mathrm{Alt}^R(D_*(M)^r, D_*(\bigwedge^r M)). \tag{2.35}$$

It follows that the isomorphism $T_{\mathrm{alt}}(D_*(M)^r) \cong D_*(\bigwedge^r M)$ given in the previous proposition is mapped to the universal alternating morphism $M^r \to \bigwedge^r M$, under the composition of the two isomorphism (2.34) and (2.35). In other words, using the isomorphism $T_{\mathrm{alt}}(D_*(M)^r) \cong D_*(\bigwedge^r M)$ we obtain an isomorphism

$$\mathrm{Alt}^R(D_*(M)^r, T_{\mathrm{alt}}(D_*(M)^r)) \cong \mathrm{Alt}^R(M^r, \bigwedge^r M)$$

and under this isomorphism, the universal elements correspond to each other. $\diamondsuit$

Lemma 2.36. *Let G be a finite R-module scheme over k of order a power of p. Denote $\bigwedge^r_{W\otimes_{\mathbb{Z}}R} D_*(G)$ by $\bigwedge^r D_*(G)$. Assume that there are commuting morphisms $\varphi : \bigwedge^r D_*(G) \to \bigwedge^r D_*(G)$ respectively $\upsilon : \bigwedge^r D_*(G) \to \bigwedge^r D_*(G)$ which are ${}^{\sigma^{-1}}\otimes\mathrm{Id}$ respectively ${}^{\sigma}\otimes\mathrm{Id}$-linear and make the F-diagram associated to φ and the V-diagram associated to υ commute. Assume further that every element of $\bigwedge^r D_*(G)$ is a linear combination of elements of the form $x_1 \wedge V x_2 \wedge \cdots \wedge V x_r$. Then $\bigwedge^r D_*(G)$ is the covariant Dieudonné module of $\bigwedge^r G$ with F and V acting respectively through φ and υ respectively.*

Proof. We know that $D_*(G)$ is a finite length W-module, and therefore $\bigwedge^r D_*(G)$ is a finite length module over $W \otimes_{\mathbb{Z}} R$. The lemma now follows from Lemma 2.30 and Proposition 2.32. $\square$

Remark 2.37. It follows from this lemma that the universal morphism $D_*(G)^r \to D_*(\bigwedge^r G) \cong \bigwedge^r D_*(G)$ (cf. Remarks 2.29 and 2.33) is the "natural" one, i.e., the one sending $(x_1, \ldots, x_r)$ to $x_1 \wedge \cdots \wedge x_r$. $\diamondsuit$

Lemma 2.38. *Let G_1 and G_2 be finite R-module schemes over k, of order a power of p with G_1 étale. Then the tensor product $G_1 \otimes G_2$ exist and its Dieudonné module, $D_*(G_1 \otimes G_2)$, is canonically isomorphic to the tensor product $D_*(G_1) \otimes_{W\otimes_{\mathbb{Z}}R} D_*(G_2)$, of Dieudonné modules of G_1 and G_2.*

Proof. The modules of $D_*(G_1)$ and $D_*(G_2)$ are of finite length over $W \otimes_{\mathbb{Z}} R$ and G_1 being étale, its Verschiebung is an isomorphism. The lemma now follows from Lemma 2.31 and Proposition 2.32. $\square$

3 Multilinear Theory of π-Divisible Modules

In this section, $\mathcal{O}$ denotes the ring of integers of a non-Archimedean local field K, with a fixed uniformizer π and finite residue field $\mathbb{F}_q$ ($q = p^f$). If the characteristic of K is p, we identify $\mathcal{O}$ with $\mathbb{F}_q[\![\pi]\!]$. We will work with *π-divisible modules*. These are natural generalizations of p-divisible groups and they enjoy similar properties. We refer to Appendix B for the definitions and basic properties of π-divisible modules. For a π-divisible module $\mathcal{M}$, we denote by $\mathcal{M}_n$ the kernel of π^n. We prove the main result of the paper, namely, that the exterior powers of π-divisible modules of dimension at most one over fields exist. We also calculate the height and dimension of these exterior powers.

3.1 Exterior powers

Definition 3.1. Let $\mathcal{M}_0, \ldots, \mathcal{M}_r, \mathcal{M}$ and $\mathcal{N}$ be π-divisible modules over a scheme S.

(i) An *$\mathcal{O}$-multilinear morphism* $\varphi : \mathcal{M}_1 \times \cdots \times \mathcal{M}_r \to \mathcal{M}_0$ is a system of $\mathcal{O}$-multilinear morphisms $\{\varphi_n : \mathcal{M}_{1,n} \times \cdots \times \mathcal{M}_{r,n} \to \mathcal{M}_{0,n}\}_n$ over S, compatible with the projections $\pi. : \mathcal{M}_{i,n+1} \twoheadrightarrow \mathcal{M}_{i,n}$ and $\pi. : \mathcal{M}_{0,n+1} \twoheadrightarrow \mathcal{M}_{0,n}$. In other words, it is an element of the inverse limit

$$\varprojlim_n \mathrm{Mult}_S^{\mathcal{O}}(\mathcal{M}_{1,n} \times \cdots \times \mathcal{M}_{r,n}, \mathcal{M}_{0,n}),$$

with transition homomorphisms induced by the projections $\pi. : \mathcal{M}_{i,n+1} \twoheadrightarrow \mathcal{M}_{i,n}$. Denote the group of $\mathcal{O}$-multilinear morphisms from $\mathcal{M}_1 \times \cdots \times \mathcal{M}_r$ to $\mathcal{M}_0$ by $\mathrm{Mult}_S^{\mathcal{O}}(\mathcal{M}_1 \times \cdots \times \mathcal{M}_r, \mathcal{M}_0)$.

(ii) We define *alternating* $\mathcal{O}$-multilinear morphism $\varphi : \mathcal{N}^r \to \mathcal{M}$ similarly and denote the group of alternating $\mathcal{O}$-multilinear morphisms from $\mathcal{N}^r$ to $\mathcal{M}$ by $\mathrm{Alt}_S^{\mathcal{O}}(\mathcal{N}^r, \mathcal{M})$. ▲

Definition 3.2. Let $\mathcal{M}, \mathcal{M}'$ be π-divisible modules over S. An alternating $\mathcal{O}$-multilinear morphism $\lambda : \mathcal{M}^r \to \mathcal{M}'$, or by abuse of terminology, the π-divisible module $\mathcal{M}'$, is called an r^{th} *exterior power* of $\mathcal{M}$ over $\mathcal{O}$, if for all π-divisible module $\mathcal{N}$ over S, the induced morphism

$$\lambda^* : \mathrm{Hom}_S(\mathcal{M}', \mathcal{N}) \to \mathrm{Alt}_S^{\mathcal{O}}(\mathcal{M}^r, \mathcal{N}), \quad \psi \mapsto \psi \circ \lambda,$$

is an isomorphism. If such $\mathcal{M}'$ and λ exist, we write $\bigwedge_{\mathcal{O}}^r \mathcal{M}$ (or $\bigwedge^r \mathcal{M}$ if no confusion is likely) for $\mathcal{M}'$ and call λ *the universal alternating morphism* defining $\bigwedge_{\mathcal{O}}^r \mathcal{M}$. ▲

3.2 The main theorem: the étale case

The category of finite étale group schemes is equivalent to the category of finite Abelian groups with a continuous action of the étale fundamental group of the base. Also, under this equivalence, finite étale $\mathcal{O}$-module schemes correspond to finite $\mathcal{O}$-modules with a continuous action. This "dictionary" allows us to construct the tensor objects and in particular the exterior powers of finite étale $\mathcal{O}$-module schemes and π-divisible modules. This is what we do in this subsection.

Proposition 3.3. *Let S be a base scheme. Let H (resp. G) be a finite étale $\mathcal{O}$-module scheme (resp. étale π-divisible module of height h) over S. Then there exists a finite étale $\mathcal{O}$-module scheme $\bigwedge^r H$ (resp. étale π-divisible group $\bigwedge^r G$ of height $\binom{h}{r}$) over S and an alternating morphism $\lambda : H^r \to \bigwedge^r H$ (resp. $\lambda : G^r \to \bigwedge^r G$) such that for all $\mathcal{O}$-module schemes X (resp. π-divisible module Y) over S the induced homomorphism*

$$\lambda^* : \mathrm{Hom}_S^{\mathcal{O}}(\bigwedge^r H, X) \to \mathrm{Alt}_S^{\mathcal{O}}(H^r, X) \quad (\textit{resp. } \lambda^* : \mathrm{Hom}_S^{\mathcal{O}}(\bigwedge^r G, Y) \to \mathrm{Alt}_S^{\mathcal{O}}(G^r, Y))$$

is an isomorphism. Furthermore, if T is an S-scheme, then the canonical homomorphism $\bigwedge^r(H_T) \to (\bigwedge^r H)_T$ (resp. $\bigwedge^r(G_T) \to (\bigwedge^r G)_T$), induced by the universal property of $\bigwedge^r(H_T)$ (resp. $\bigwedge^r(G_T)$) and λ_T, is an isomorphism. In other words, the exterior powers of H (resp. G) exist, are again étale and their construction commutes with arbitrary base change.

Proof. We prove the two cases (finite étale and π-divisible module) separately.

- Assume at first that S is connected and choose a geometric point $\bar{s}$ of S. Then the functor

$$J \mapsto J(\bar{s}) = \mathrm{Mor}_S(\bar{s}, J)$$

from the category of finite étale $\mathcal{O}$-module schemes over S to the category of finite $\mathcal{O}$-modules with a continuous action of the étale fundamental group at $\bar{s}$, $\pi_1^{\text{ét}}(S, \bar{s})$, is an equivalence of categories. Moreover, this functor preserves multilinear and alternating morphisms. The category of $\mathcal{O}$-modules with a continuous $\pi_1^{\text{ét}}(S, \bar{s})$-action is a tensor category and has in particular all exterior powers. It follows that the category of finite étale $\mathcal{O}$-module schemes over S possesses all exterior powers over S. The universal alternating morphism $H(\bar{s})^r \to \bigwedge^r(H(\bar{s}))$ induces the universal alternating morphism $\lambda : H^r \to \bigwedge^r H$ with the universal property stated in the proposition. If S is not connected, the finite étale group scheme H decomposes into disjoint union of finite étale $\mathcal{O}$-module schemes over connected components of S. The above construction yields exterior powers of each of them over a connected base and these exterior powers glue together to produce exterior powers of the whole $\mathcal{O}$-module scheme H. One has to verify that these exterior powers satisfy the universal property of exterior powers, but this is true since homomorphisms and multilinear morphisms of $\mathcal{O}$-module schemes over S decompose into homomorphisms and multilinear morphisms of $\mathcal{O}$-module schemes over each connected component. This proves the existence of the exterior powers. Now assume that T is an S-scheme. For the statement on the base change, we can assume that S and T are both connected. Fix a geometric point $\bar{t}$ of T lying over the fixed geometric point $\bar{s}$ of S. The structural morphism $f : T \to S$ induces an $\mathcal{O}$-module homomorphism $f_* : \pi_1^{\text{ét}}(T, \bar{t}) \to \pi_1^{\text{ét}}(S, \bar{s})$. For every finite étale scheme J over S, the map of sets $J(\bar{s}) \to J_T(\bar{t})$ is a bijection and the action of $\pi_1^{\text{ét}}(T, \bar{t})$ on $J_T(\bar{t})$ is the restriction, under the homomorphism f_*, of the action of $\pi_1^{\text{ét}}(S, \bar{s})$ on $J(\bar{s})$, after identifying the two sets $J(\bar{s})$ and $J_T(\bar{t})$. In other words, the base extension from S to T of étale group schemes over S corresponds to the restriction of the action of the étale fundamental group of S at $\bar{s}$ to that of T at $\bar{t}$. It follows at once that the $\mathcal{O}$-module scheme homomorphism $\bigwedge^r(H_T) \to (\bigwedge^r H)_T$ is an isomorphism.

- Similarly, assuming S is connected and fixing a geometric point $\bar{s}$ of S, the functor that assigns to an étale π-divisible module its Tate module (at $\bar{s}$) defines an equivalence of categories between the category of étale π-divisible module over S and the category of finite free $\mathcal{O}$-modules with a continuous action of the étale fundamental group of S at $\bar{s}$. Also, Multilinear (respectively alternating) morphisms are preserved under this functor. The category of finite free $\mathcal{O}$-modules with a continuous $\pi_1^{\text{ét}}(S, \bar{s})$-action is a tensor category and therefore possesses all exterior powers. It implies that the category of étale π-divisible modules over S has all exterior powers in the sense stated in the proposition, with the universal alternating morphism $\lambda : G^r \to \bigwedge^r G$ obtained from the universal alternating morphism $T_p(G)^r \to \bigwedge^r T_p(G) \cong T_p(\bigwedge^r G)$. The height of an étale π-divisible group is equal to the rank over $\mathcal{O}$ of its Tate module. Thus, the height of $\bigwedge^r G$ is equal to the rank over $\mathcal{O}$ of $T_p(\bigwedge^r G) \cong \bigwedge^r(T_p(G))$, which is equal to $\binom{h}{r}$. The same arguments as in the last case (of finite étale $\mathcal{O}$-module schemes) show that this construction commutes with arbitrary base change.

$\square$

Remark 3.4. If $\mathcal{M}$ is étale, then for every positive natural number n, we have $\bigwedge^r(\mathcal{M}_n) \cong (\bigwedge^r \mathcal{M})_n$. $\Diamond$

3.3 The main theorem: over fields of characteristic p

In this subsection, unless otherwise specified, k is a perfect field of characteristic p.

Construction 3.5. Let $\mathcal{M}_0, \mathcal{M}_1, \ldots, \mathcal{M}_r$ be π-divisible modules over k and for every $i = 0, \ldots, r$, denote by D_i the Dieudonné module of $\mathcal{M}_i$. Let $f : D_1 \times \cdots \times D_r \to D_0$ be an $\mathcal{O}$-multilinear morphism (of $W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}$-modules) satisfying the V-condition, i.e., $Vf(x_1, \ldots, x_r) = f(Vx_1, \ldots, Vx_r)$ for every $x_i \in D_i$. Since by Lemma B.13, V is injective, Remark 1.19 applies and so, $f \in \mathrm{Mult}^{\mathcal{O}}(D_1 \times \cdots \times D_r, D_0)$ (cf. Definition 1.18). For every $n \geq 1$, this morphism induces a morphism $D_1 \times \cdots \times D_r \to D_0/\pi^n D_0$, and using the multilinearity of f, we obtain an $\mathcal{O}$-multilinear morphism

$$D_1/\pi^n D_1 \times \cdots \times D_r/\pi^n D_r \to D_0/\pi^n D_0$$

that we denote by f_n. It follows from its construction, that f_n belongs to $\mathrm{Mult}^{\mathcal{O}}(D_{1,n} \times \cdots \times D_{r,n}, D_{0,n})$, where we denote by $D_{i,n}$ the Dieudonné module of $\mathcal{M}_{i,n} = \mathrm{Ker}(\pi^n : \mathcal{M}_i \to \mathcal{M}_i)$, which is canonically isomorphic to $D_i/\pi^n D_i$. Thus, the construction of f_n from f defines an $\mathcal{O}$-linear morphism

$$\alpha_n : \mathrm{Mult}^{\mathcal{O}}(D_1 \times \cdots \times D_r, D_0) \to \mathrm{Mult}^{\mathcal{O}}(D_{1,n} \times \cdots \times D_{r,n}, D_{0,n}).$$

These morphisms are compatible with the canonical morphisms

$$\mathrm{Mult}^{\mathcal{O}}(D_{1,n+1} \times \cdots \times D_{r,n+1}) \to \mathrm{Mult}^{\mathcal{O}}(D_{1,n} \times \cdots \times D_{r,n}, D_{0,n})$$

given by the projections $D_{i,n+1} \to D_{i,n}$ and therefore define an $\mathcal{O}$-linear morphism

$$\alpha : \mathrm{Mult}^{\mathcal{O}}(D_1 \times \cdots \times D_r, D_0) \to \varprojlim_n \mathrm{Mult}^{\mathcal{O}}(D_{1,n} \times \cdots \times D_{r,n}).$$

Similarly, α restricts to an $\mathcal{O}$-linear morphism $\mathrm{Alt}^{\mathcal{O}}(D_1^r, D_0) \to \varprojlim_n \mathrm{Alt}^{\mathcal{O}}(D_{1,n}^r, D_{0,n})$. ▼

Lemma 3.6. *Let $\mathcal{M}_0, \mathcal{M}_1, \dots, \mathcal{M}_r$ be π-divisible modules over k. The $\mathcal{O}$-linear morphisms*

$$\alpha : \mathrm{Mult}^{\mathcal{O}}(D_1 \times \cdots \times D_r, D_0) \to \varprojlim_n \mathrm{Mult}^{\mathcal{O}}(D_{1,n} \times \cdots \times D_{r,n}, D_{0,n})$$

and

$$\mathrm{Alt}^{\mathcal{O}}(D_1^r, D_0) \to \varprojlim_n \mathrm{Alt}^{\mathcal{O}}(D_{1,n}^r, D_{0,n})$$

constructed above are isomorphisms.

Proof. Let us at first show that α is an isomorphism. Define a map

$$\omega : \varprojlim_n \mathrm{Mult}^{\mathcal{O}}(D_{1,n} \times \cdots \times D_{r,n}, D_{0,n}) \to \mathrm{Mult}^{\mathcal{O}}(D_1 \times \cdots \times D_r, D_0)$$

as follows. Take an element $g = (g_n)_n$ in the inverse limit. By assumption, the following diagram commutes:

$$\begin{tikzcd} D_{1,n+1} \times \cdots \times D_{r,n+1} \arrow[r, "g_{n+1}"] \arrow[d] & D_{0,n+1} \arrow[d] \\ D_{1,n} \times \cdots \times D_{r,n} \arrow[r, "g_n"'] & D_{0,n} \end{tikzcd}$$

where the vertical morphisms are the canonical projections. Set

$$\omega(g) : D_1 \times \cdots \times D_r \to D_0, \quad \big((u_{1,j})_j, (u_{2,j})_j, \dots, (u_{r,j})_j\big) \mapsto \big(g_j(u_{1,j}, \dots, u_{r,j})\big)_j,$$

where $(u_{i,j})_j$ is an element of $D_i = \varprojlim_j D_{i,j}$. The commutativity of the above diagram implies that $\big(g_j(u_{1,j}, \dots, u_{r,j})\big)_j$ is an element of the inverse limit $\varprojlim_j D_{0,j} = D_0$, and by construction, $\omega(g)$ satisfies the V-condition and by the above discussion (see Remark 1.19) satisfies the F-conditions as well. It is straightforward to see that the compositions $\alpha \circ \omega$ and $\omega \circ \alpha$ are identities. If $\mathcal{M}_1 = \mathcal{M}_2 = \cdots = \mathcal{M}_r$ and for all $n \geq 1$, g_n is alternating, then $\omega(g)$ is alternating which implies that the restriction of α to $\mathrm{Alt}^{\mathcal{O}}(D_1^r, D_0)$ induces an isomorphism $\mathrm{Alt}^{\mathcal{O}}(D_1^r, D_0) \to \varprojlim_n \mathrm{Alt}^{\mathcal{O}}(D_{1,n}^r, D_{0,n})$. ☐

Corollary 3.7. *Let $\mathcal{M}_0, \mathcal{M}_1, \dots, \mathcal{M}_r$ be π-divisible modules over k. For every $i = 0, \dots, r$, denote by D_i the (covariant) Dieudonné module of $\mathcal{M}_i$. There exist natural functorial isomorphisms*

$$\mathrm{Mult}^{\mathcal{O}}(D_1 \times \cdots \times D_r, D_0) \cong \mathrm{Mult}_k^{\mathcal{O}}(\mathcal{M}_1 \times \cdots \times \mathcal{M}_r, \mathcal{M}_0)$$

and

$$\mathrm{Alt}^{\mathcal{O}}(D_1^r, D_0) \cong \mathrm{Alt}_k^{\mathcal{O}}(\mathcal{M}_1^r, \mathcal{M}_0).$$

Proof. This follows from Corollary 1.23 and Remark 1.24 (by taking inverse limits) and last lemma. ☐

Notations 3.8. Let us fix some notations for the rest of this subsection.

- We fix a natural number r.
- Unless otherwise specified, $p > 2$.
- $\mathcal{M}$ is a π-divisible module of dimension 1 and height h over k.
- W is the ring of Witt vectors over k and L is the fraction field of W.
- $D := D(\mathcal{M})$ is the covariant Dieudonné module of $\mathcal{M}$ and $\bigwedge^r D := \bigwedge^r_{W\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O}} D$.
- $\forall i$, $D_i := D_*(\mathcal{M}_i)$ is the covariant Dieudonné module of $\mathcal{M}_i$ and $\bigwedge^r D_i := \bigwedge^r_{W\otimes_{\mathbb{Z}_p}\frac{\mathcal{O}}{\pi^i}} D_i$.
- Denote by ζ the surjection $\zeta : D \twoheadrightarrow D_i$ and by $\bigwedge^r \zeta$ the surjection $\bigwedge^r D \twoheadrightarrow \bigwedge^r D_i$. Note that ζ doesn't have any index (to avoid complexity) and we use the same letter for different indices.
- Denote by Υ (respectively υ) the morphism $\bigwedge^r V : \bigwedge^r D \to \bigwedge^r D$ (respectively $\bigwedge^r V : \bigwedge^r D_i \to \bigwedge^r D_i$) sending an element $d_1 \wedge \cdots \wedge d_r$ with $d_1, \cdots, d_r \in D$ (respectively in D_i) to $Vd_1 \wedge \cdots \wedge Vd_r$.

Remark 3.9. Note that we have $\bigwedge^r \zeta \circ \Upsilon = \upsilon \circ \bigwedge^r \zeta$. $\Diamond$

Lemma 3.10. *Assume that k is algebraically closed.*

a) There exist a $\mathbb{Z}_p$-algebra A and a decomposition $W \otimes_{\mathbb{Z}_p} \mathcal{O} = \prod_{\mathbb{Z}/f\mathbb{Z}} W \otimes_A \mathcal{O}$, such that $W \otimes_A \mathcal{O}$ is a discrete valuation ring with residue field k and maximal ideal generated by $1 \otimes \pi$.

b) The decomposition in a) gives the a decomposition $W\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O} = \prod_{\mathbb{Z}/f\mathbb{Z}} W\widehat{\otimes}_A\mathcal{O}$ of the completed tensor product $W\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O}$ as a product of complete discrete valuation rings with maximal ideal generated by $1\widehat{\otimes}\pi$ and residue field equal to k.

c) Let N be a $W\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O}$-module endowed with a ${}^{\sigma}$-linear morphism $\varphi : N \to N$, i.e., for every $x \in W\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O}$ and $n \in N$, we have $\varphi(x \cdot n) = ({}^{\sigma}\widehat{\otimes}\,\mathrm{Id})(x) \cdot \varphi(n)$. Then there is a decomposition of N as a product $\prod_{i\in\mathbb{Z}/f\mathbb{Z}} N_i$ into $W\widehat{\otimes}_A\mathcal{O}$-modules, according to the decomposition of $W\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O}$ given above, such that the morphism φ restricts to morphisms $\varphi : N_i \to N_{i-1}$ for all $i \in \mathbb{Z}/f\mathbb{Z}$.

Proof.

a) We prove this lemma in equal and mixed characteristic cases separately.

- Equal characteristic: Set $A := \mathbb{F}_q$ and let $\mathbb{Z}_p \to \mathbb{F}_q$ be the canonical ring homomorphism. In this case, $\mathcal{O}$ is isomorphic to $\mathbb{F}_q[\![\pi]\!]$ and therefore, the tensor product $W \otimes_{\mathbb{Z}_p} \mathcal{O}$ is isomorphic to $k \otimes_{\mathbb{F}_p} \mathbb{F}_q[\![\pi]\!]$ which decomposes as $\prod_{\mathbb{Z}/f\mathbb{Z}} k \otimes_{\mathbb{F}_q} \mathbb{F}_q[\![\pi]\!] = \prod_{\mathbb{Z}/f\mathbb{Z}} k[\![\pi]\!]$ (cf. the proof of Theorem B.14 for more details). It is then clear that the ring $k[\![\pi]\!]$ is a discrete valuation ring with residue field k and maximal ideal generated by π.
- Mixed characteristic: Let E be the maximal unramified subextension of $K = \mathrm{Frac}(\mathcal{O})$ and denote by A its ring of integers. We then have a canonical ring extension $\mathbb{Z}_p \hookrightarrow A$. Since k is algebraically closed, there is a copy of A inside W. As E is the maximal unramified subextension of K, the degree of the extension $E/\mathbb{Q}_p$ is equal to f and therefore we have an A-algebra isomorphism $W \otimes_{\mathbb{Z}_p} A \cong \prod_{\mathbb{Z}/f\mathbb{Z}} W \otimes_A A = \prod_{\mathbb{Z}/f\mathbb{Z}} W$ given on elements by $w \otimes a \mapsto (a \cdot w^{\sigma^i})_i$, where ${}^{\sigma} : A \to A$ is the Frobenius of A, induced by the Frobenius of W. It follows that $W \otimes_{\mathbb{Z}_p} \mathcal{O} \cong W \otimes_{\mathbb{Z}_p} A \otimes_A \mathcal{O} \cong \prod_{\mathbb{Z}/f\mathbb{Z}} W \otimes_A \mathcal{O}$ and the Frobenius, i.e., the morphism ${}^{\sigma}\otimes\mathrm{Id}$ interchanges the factors. This shows the first statement. Now, as K is totally ramified over E, $\mathcal{O}$ is generated over A by an Eisenstein element and since L is unramified over E, the same element is again Eisenstein over W. Hence, $L \otimes_E K$ is a field and $W \otimes_A \mathcal{O}$ is the valuation ring in it. Again, since $E/\mathbb{Q}_p$ is the maximal unramified extension inside K,

the residue degree of the extension K/E is one and therefore $\mathcal{O}$ and A have the same residue fields $\mathbb{F}_q$. Therefore, we have

$$\frac{W \otimes_A \mathcal{O}}{(1 \otimes \pi)W \otimes_A \mathcal{O}} = W \otimes_A \mathcal{O}/\pi \cong W \otimes_{\mathbb{F}_q} \mathbb{F}_q = k$$

where the first equality follows from flatness of W over $\mathbb{Z}_p$. This proves the other statement.

b) Since for every $s > 0$, $p^s\mathcal{O} \subset \pi^{s'}\mathcal{O}$ for some s', the submodule $p^s \otimes \mathcal{O} + W \otimes \pi^r\mathcal{O}$ of $W \otimes_{\mathbb{Z}_p} \mathcal{O}$ is equal to $W \otimes \pi^{r'}\mathcal{O}$ for some r', and therefore the completed tensor product $W\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O}$ is equal to

$$\varprojlim_r \frac{W \otimes_{\mathbb{Z}_p} \mathcal{O}}{W \otimes_{\mathbb{Z}_p} \pi^r\mathcal{O}} = \varprojlim W \otimes_{\mathbb{Z}_p} \frac{\mathcal{O}}{\pi^r}$$

where the last equality follows from flatness of W over $\mathbb{Z}_p$. Now using part *a*), we have

$$\varprojlim(W \otimes_{\mathbb{Z}_p} \frac{\mathcal{O}}{\pi^r}) = \varprojlim \prod W \otimes_A \frac{\mathcal{O}}{\pi^r} = \prod \varprojlim(W \otimes_A \frac{\mathcal{O}}{\pi^r}).$$

As we have seen in *a*), $W \otimes_A \mathcal{O}$ is a discrete valuation ring with uniformizer $1 \otimes \pi$, and this implies that $\varprojlim W \otimes_A \frac{\mathcal{O}}{\pi^r}$ is the $1 \otimes \pi$-adic completion of it, which is a complete discrete valuation ring with uniformizer $1\widehat{\otimes}\pi$ and residue field equal to k.

c) Let e_i be the primitive idempotent for the i^{th} factor of the decomposition of $W\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O}$. This decomposition gives a decomposition of N, say $N = \prod N_i$, where $N_i = e_iN$. Now, for any $n \in N$, we have $\varphi(e_in) = ({}^{\sigma}\widehat{\otimes}\,\mathrm{Id})(e_i)\varphi(n) \subset e_{i-1}N = N_{i-1}$. Hence, $\varphi(N_i) \subset N_{i-1}$.

□

Remark 3.11. 1) The proof of part *a*) of the lemma in the mixed characteristic case is inspired by the proof of the lemma in [8].

2) Part *b*) of the lemma implies that $\frac{W\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O}}{(1\widehat{\otimes}\pi^i)W\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O}} = \prod \frac{W\widehat{\otimes}_A\mathcal{O}}{(1\widehat{\otimes}\pi^i)W\widehat{\otimes}_A\mathcal{O}} = \prod W \otimes_A \frac{\mathcal{O}}{\pi^i} = W \otimes_{\mathbb{Z}_p} \frac{\mathcal{O}}{\pi^i}$. ◊

Lemma 3.12. *Assume that k is algebraically closed. The Dieudonné module of $\mathcal{M}$, is a free $W\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O}$-module of rank h. If $\mathcal{M}$ is connected, then there exists an element $\varepsilon \in D$ such that $\{\varepsilon, V^f\varepsilon, \cdots, V^{(h-1)f}\varepsilon\}$ is a basis of D over $W\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O}$.*

Proof. From Lemma 3.10 *c*), we know that the Dieudonné module has a decomposition $D = \prod_{i\in\mathbb{Z}/f\mathbb{Z}} M_i$, where each M_i is a module over $W\widehat{\otimes}_A\mathcal{O}$ and that the Verschiebung permutes them cyclically (since it is σ^{-1}-linear). We want to show that each M_i is a free $W\widehat{\otimes}_A\mathcal{O}$-module of rank h. The Dieudonné module D_1 is a W-module of finite length, which implies that it is a finite length module over $W \otimes_A \mathcal{O}$, where A is the ring defined in Lemma 3.10, but $\pi D_1 = 0$ and therefore D_1 is a module of finite length over $W \otimes_A \mathcal{O}/\pi = k$ and we know that its length is $\log_p |\mathcal{M}_1| = fh$. Take fh elements in D such that their images in D_1 generate D_1 over k, then by Nakayama lemma, they generate D over $W\widehat{\otimes}_A\mathcal{O}$. Note that the action of π on D is free, this follows from the fact that the kernel of π on each D_i is the same module D_1, and the transition morphisms from D_{i+1} to D_i is multiplication by π, and therefore the kernel of π on D is the inverse limit of D_1 with trivial transition morphisms, and hence it is trivial. Therefore D is a finitely generated torsion-free $W\widehat{\otimes}_A\mathcal{O}$-module and since by Lemma 3.10 *b*) $W\widehat{\otimes}_A\mathcal{O}$ is a discrete valuation ring, D is free over $W\widehat{\otimes}_A\mathcal{O}$. The rank of D over $W\widehat{\otimes}_A\mathcal{O}$ is equal to the length of D_1 over $W \otimes_A \mathcal{O}/\pi = k$, which is fh. It follows that M_i are free $W\widehat{\otimes}_A\mathcal{O}$-modules of finite rank. As $V : D \to D$ is injective by Lemma B.13, and its restriction to M_i is a morphism $M_i \to M_{i+1}$ (for all $i \in \mathbb{Z}/f\mathbb{Z}$), the M_i will all have the same rank h over $W\widehat{\otimes}_A\mathcal{O}$. This shows that D is free of rank h.

Now, assume that $\mathcal{M}$ is connected. If we can find elements $\varepsilon_i \in M_i$ ($i \in \mathbb{Z}/f\mathbb{Z}$) such that the set $\{\varepsilon_i, V^f\varepsilon_i, \cdots, V^{(h-1)f}\varepsilon_i\}$ is a basis of M_i over $W\widehat{\otimes}_A\mathcal{O}$ (note that $V^{jf}(M_i) \subset M_i$ for every $j \geq 0$), then the element $\varepsilon := \varepsilon_1 + \varepsilon_2 \cdots + \varepsilon_f$ will be the desired element and we are done. Since $W\widehat{\otimes}_A\mathcal{O}$ is a local ring with maximal ideal generated by $1\widehat{\otimes}\pi$ and since M_i is a free module of rank h over it, in order to find ε_i, it suffices (by Nakayama's lemma) to find an element $\overline{\varepsilon_i} \in \mathrm{Coker}(\pi : M_i \to M_i) := \overline{M}_i$ such that the set $\{\overline{\varepsilon_i}, V^f\overline{\varepsilon_i}, \cdots, V^{(h-1)f}\overline{\varepsilon_i}\}$ is a basis of $\overline{M}_i$ over $W\widehat{\otimes}_A\mathcal{O}/\pi \cong k$ (M_i being free of rank h over $W\widehat{\otimes}_A\mathcal{O}$, we have that $\overline{M}_i$ has dimension h over k) and then define ε_i to be a lift of $\overline{\varepsilon_i}$ in M_i. From the definition of $\overline{M}_i$ we have that $D_1 = \prod_i \overline{M}_i$ and that Verschiebung is a morphism $V : \overline{M}_i \to \overline{M}_{i+1}$. Since the dimension of $\mathcal{M}$ is 1, the Hopf algebra of $\mathcal{M}_1$ is isomorphic to $\dfrac{k[x]}{(x^{q^h})}$ (cf. [9] p.112, §14.4, Theorem), and so $F^s_{\mathcal{M}_1} = 0$ if and only if $s \geq fh$. It follows that $V^s : D_1 \to D_1$ is the zero morphism if and only if $s \geq fh$. Set $\varphi := V^f$. As stated above, we have $\varphi(\overline{M}_i) \subset \overline{M}_i$, and so we have a ${}^{\sigma^{-f}}$-linear morphism $\varphi : \overline{M}_i \to \overline{M}_i$. We claim that $\varphi^{h-1} : \overline{M}_i \to \overline{M}_i$ is not the zero morphism. Indeed, if we have $V^{(h-1)f}|_{\overline{M}_i} = \varphi^{h-1}|_{\overline{M}_i} = 0$ for some i, then for every j and every element $x \in \overline{M_j}$, we have $V^{\overline{i-j}}(x) \in \overline{M}_i$, where $\overline{i-j} \geq 0$ is the class of $i-j$ modulo f and so $V^{(h-1)f+\overline{(i-j)}}(x) = 0$. But $(h-1)f + \overline{(i-j)} < hf$. This implies that V^{hf-1} is the zero morphism on D_1, which is in contradiction with what we said above. Now, let $\overline{\varepsilon_i} \in \overline{M}_i$ be an element with $\varphi^{h-1}(\overline{\varepsilon_i}) \neq 0$. Then the set $\{\overline{\varepsilon_i}, \varphi(\overline{\varepsilon_i}), \cdots, \varphi^{h-1}(\overline{\varepsilon_i})\}$ is linearly independent over k, for if we have a non-trivial relation $\sum_{j=j_0}^{h-1} a_j\varphi^j(\overline{\varepsilon_i}) = 0$ with $a_j \in k$ and $a_{j_0} \neq 0$, then

$$0 = \varphi^{(h-1-j_0)}\Big(\sum_{j=j_0}^{h-1} a_j\varphi^j(\overline{\varepsilon_i})\Big) = \sum_{j=j_0}^{h-1} a_j^{q^{h-1-j_0}} \varphi^{h-1-j_0+j}(\overline{\varepsilon_i}) = a_{j_0}^{q^{h-1-j_0}} \varphi^{h-1}(\overline{\varepsilon_i}),$$

because $\varphi^s = 0$ for $s \geq h$. But $\varphi^{h-1}(\overline{\varepsilon_i})$ is not zero, and so $a_{j_0} = 0$, which is in contradiction with the choice of j_0. As the dimension of $\overline{M}_i$ over k is h and the set $\{\overline{\varepsilon_i}, \varphi(\overline{\varepsilon_i}), \cdots, \varphi^{h-1}(\overline{\varepsilon_i})\}$ is linearly independent, we deduce that this set is in fact a basis of $\overline{M}_i$ over k and the proof is achieved. □

Remark 3.13. 1) Note that the first part of the Lemma, i.e., that the Dieudonné module is free of rank h, is true without assuming that $\mathcal{M}$ has dimension 1.

2) Since D_i is the cokernel of π^i on D and the projection from D to D_i commutes with V, it follows from Lemma 3.12 that D_i is a free $W \otimes_{\mathbb{Z}_p} \frac{\mathcal{O}}{\pi^i}$-module of rank h and that the set $\{\bar{\varepsilon}, V^f\bar{\varepsilon}, \cdots, V^{(h-1)f}\bar{\varepsilon}\}$, where $\bar{\varepsilon}$ is the image of ε in D_i, is a basis of D_i over $W \otimes_{\mathbb{Z}_p} \frac{\mathcal{O}}{\pi^i}$.

3) In the above proof, let i be such that restriction of V^{hf-1} to $\overline{M}_i$ is not zero and choose $\varepsilon_i \in \overline{M}_i$ with $V^{hf-1}(\varepsilon_i) \neq 0$. Then for every $0 \leq j \leq f-1$, we have $V^{(h-1)f}(V^j\varepsilon_i) \neq 0$. Since for these j, we have $V^j\varepsilon_i \in \overline{M}_{i+j}$, we see that we could take ε_{i+j} to be $V^j\varepsilon_i$. This shows that we have a sequence of ${}^{\sigma^{-1}}$-linear isomorphisms $\overline{M}_i \xrightarrow[\cong]{V} \overline{M}_{i+1} \xrightarrow[\cong]{V} \overline{M}_{i+2} \xrightarrow[\cong]{V} \dots \xrightarrow[\cong]{V} \overline{M}_{i-1}$. By Nakayama's lemma, and the fact that V is injective on D, we conclude that V induces ${}^{\sigma^{-1}}$-linear isomorphisms $V : M_j \to M_{j+1}$ for every $j \neq i-1$. It follows that

$$VD \cong VM_{f-1} \times VM_0 \times VM_1 \times \cdots \times VM_{f-2} \cong M_0 \times M_1 \times \cdots \times M_{i-1} \times VM_{i-1} \times M_{i+1} \times \cdots \times M_{f-1}.$$

Thus, the Lie algebra of $\mathcal{M}$ is isomorphic to $D/VD \cong M_i/VM_{i-1} \cong M_i/V^fM_i$. ◊

Lemma 3.14. *Assume that $\mathcal{M}$ is connected. Then the morphism $\Upsilon : \bigwedge^r D \to \bigwedge^r D$ is injective.*

Proof. Since the extension $k \hookrightarrow \bar{k}$ is faithfully flat, we may assume that k is algebraically closed. A semi-linear endomorphism of a free module of finite rank is injective if and only if its determinant is a non-zero divisor. As D is a free $W\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O}$-module of rank h, $\bigwedge^r D$ is a free $W\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O}$-module of rank $\binom{h}{r}$. The determinant of $\Upsilon = \bigwedge^r V$ is equal to $\det(V)^{\binom{h-1}{r-1}}$. It follows that V is injective if and only if Υ is injective, and since by Lemma B.13, V is injective, Υ is injective too. □

Remark 3.15. Assume that $\mathcal{M}$ is connected. Then, by previous lemma, there exists at most one ${}^{\sigma^{-1}} \otimes \mathrm{Id}$-linear morphism $\Phi : \bigwedge^r D \to \bigwedge^r D$ such that $\Upsilon \circ \Phi = p$. ◊

Definition 3.16. Assume that $\mathcal{M}$ is connected and k is algebraically closed.

(i) Define a ${}^{\sigma^{-1}} \otimes \mathrm{Id}$-linear morphism $\Phi : \bigwedge^r D \to \bigwedge^r D$ by sending a basis element $V^{f\alpha_1}\varepsilon \wedge \cdots \wedge V^{f\alpha_r}\varepsilon$ to $FV^{f\alpha_1}\varepsilon \wedge V^{f\alpha_2 - 1}\varepsilon \wedge \cdots \wedge V^{f\alpha_r - 1}\varepsilon$.

(ii) Define $\varphi : \bigwedge^r D_i \to \bigwedge^r D_i$ similarly. ▲

Remark 3.17. Note that we have $\bigwedge^r \zeta \circ \Phi = \varphi \circ \bigwedge^r \zeta$. ◊

Lemma 3.18. *Assume that $\mathcal{M}$ is connected and k is algebraically closed.*

a) We have $\Phi \circ \Upsilon = p = \Upsilon \circ \Phi$ and the F-diagram associated to Φ and the V-diagram associated to Υ are commutative.

b) We have $\varphi \circ \upsilon = p = \upsilon \circ \varphi$ and the F-diagram associated to φ and the V-diagram associated to υ are commutative.

Proof. *a)* We first check the equality $\Upsilon \circ \Phi = p$. It is sufficient to calculate $\Upsilon \circ \Phi$ on the basis elements $V^{f\alpha_1}\varepsilon \wedge \cdots \wedge V^{f\alpha_r}\varepsilon$:

$$\Upsilon \circ \Phi(V^{f\alpha_1}\varepsilon \wedge \cdots \wedge V^{f\alpha_r}\varepsilon) = \Upsilon(FV^{f\alpha_1}\varepsilon \wedge V^{f\alpha_2 - 1}\varepsilon \wedge \cdots \wedge V^{f\alpha_r - 1}\varepsilon) =$$

$$VFV^{f\alpha_1}\varepsilon \wedge \cdots \wedge V^{f\alpha_r}\varepsilon = pV^{f\alpha_1}\varepsilon \wedge \cdots \wedge V^{f\alpha_r}\varepsilon$$

where the first and second equality follow respectively from the definition of Φ and Υ (cf. Notations 3.8), and the last equality follows from the equality $VF = p$. Hence the equality $\Upsilon \circ \Phi = p$.

Now, we calculate $\Upsilon \circ \Phi \circ (\mathrm{Id} \wedge V \wedge \cdots \wedge V)$:

$$\Upsilon \circ \Phi \circ (\mathrm{Id} \wedge V \wedge \cdots \wedge V) = p \circ (\mathrm{Id} \wedge V \wedge \cdots \wedge V) = p \wedge V \wedge \cdots \wedge V =$$

$$VF \wedge V \wedge \cdots \wedge V = \Upsilon \circ (F \wedge \mathrm{Id} \wedge \cdots \wedge \mathrm{Id})$$

where the first equality follows from the equality $\Upsilon \circ \Phi = p$ and the other equalities follow from the definition of Υ and the equality $VF = p$. But we know from Lemma 3.14 that Υ is injective and therefore, we have $\Phi \circ (\mathrm{Id} \wedge V \wedge \cdots \wedge V) = F \wedge \mathrm{Id} \wedge \cdots \wedge \mathrm{Id}$. Denoting by λ the universal alternating morphism $D \times \cdots \times D \to \bigwedge^r D$ sending an element $(x_1, \cdots, x_r)$ to $x_1 \wedge \cdots \wedge x_r$, the last equality gives rise to the following diagram:

$$\begin{tikzcd}
D \times \cdots \times D \arrow[rr, "\lambda"] & & \bigwedge^r D \arrow[dd, "\Phi"] \\
D \times \cdots \times D \arrow[u, "\mathrm{id} \times V \times \cdots \times V"] \arrow[r, "\lambda"] \arrow[d, "F \times \mathrm{Id} \times \cdots \times \mathrm{Id}"'] & \bigwedge^r D \arrow[ur, "\mathrm{Id} \wedge V \wedge \cdots \wedge V"] \arrow[dr, "F \wedge \mathrm{Id} \wedge \cdots \wedge \mathrm{Id}"'] & \\
D \times \cdots \times D \arrow[rr, "\lambda"'] & & \bigwedge^r D
\end{tikzcd}$$

with the right triangle commutative. It follows that the whole diagram is commutative and thus the F-diagram associated to Φ is commutative. The commutativity of the V-diagram associated to Υ follows from its definition (in fact it is equivalent to the definition of Υ!). It remains to show that $\Phi \circ \Upsilon = p$. We have:

$$\Phi \circ \Upsilon(x_1 \wedge \cdots \wedge x_r) = \Phi(Vx_1 \wedge \cdots \wedge Vx_r) = FVx_1 \wedge x_2 \wedge \cdots \wedge x_r = px_1 \wedge \cdots \wedge x_r$$

where the second equality follows from the V-diagram associated to Υ, the third one from the F-diagram associated to Φ and the last once, again, from the equality $FV = p$.

b) The compatibility of Υ and υ, and of Φ and φ with respect to the epimorphism $\zeta : D \twoheadrightarrow D_i$ (cf. Remarks 3.9 and 3.17) and statement *a)* of the lemma imply the following properties:

1) $\upsilon \circ \varphi \circ \bigwedge^r \zeta = \upsilon \circ \bigwedge^r \zeta \circ \Phi = \bigwedge^r \zeta \circ \Upsilon \circ \Phi = \bigwedge^r \zeta \circ p = p \circ \bigwedge^r \zeta$.
2) $\varphi \circ \upsilon \circ \bigwedge^r \zeta = \varphi \circ \bigwedge^r \zeta \circ \upsilon = \bigwedge^r \zeta \circ \Phi \circ \Upsilon = \bigwedge^r \zeta \circ p = p \circ \bigwedge^r \zeta$.
3) $\varphi \circ \lambda \circ (\mathrm{Id} \times V \times \cdots \times V) \circ \zeta^r = \varphi \circ \lambda \circ \zeta^r \circ (\mathrm{Id} \times V \times \cdots \times V) = \varphi \circ \bigwedge^r \zeta \circ \lambda \circ (\mathrm{Id} \times V \times \cdots \times V) = \bigwedge^r \zeta \circ \Phi \circ \lambda \circ (\mathrm{Id} \times V \times \cdots \times V) = \bigwedge^r \zeta \circ \lambda \circ (F \times \mathrm{Id} \times \cdots \times \mathrm{Id}) = \lambda \circ \zeta^r \circ (F \times \mathrm{Id} \times \cdots \times \mathrm{Id}) = \lambda \circ (F \times \mathrm{Id} \times \cdots \times \mathrm{Id}) \circ \zeta^r$.

Since the morphism $\zeta : D \to D_i$ is surjective, the morphisms $\bigwedge^r \zeta : \bigwedge^r D \to \bigwedge^r D_i$ and $\zeta^r : D^r \to D_i^r$ are surjective as well and thus we can cancel them from the right and conclude from properties 1) and 2) that $\upsilon \circ \varphi = p = \varphi \circ \upsilon$ and from 3) that the F-diagram associated to φ is commutative. The commutativity of the V-diagram associated to υ follows once more from the definition of υ.

□

Proposition 3.19. *Assume that k is algebraically closed. Then the Dieudonné module of $\bigwedge^r \mathcal{M}_i$ is isomorphic to $\bigwedge^r D_i$ and in particular the order of $\bigwedge^r \mathcal{M}_i$ is equal to $q^{i\binom{h}{r}}$. More precisely we have:*

- *a) if $\mathcal{M}$ is étale, then the module scheme $\bigwedge^r \mathcal{M}_i$ is isomorphic to the constant module scheme $\bigwedge^r (\frac{\mathcal{O}}{\pi^i})^h$, which has order $q^{i\binom{h}{r}}$ and*
- *b) if $\mathcal{M}$ is connected, then the covariant Dieudonné module of $\bigwedge^r \mathcal{M}_i$ is isomorphic to $\bigwedge^r D_i$ with the actions of F respectively V defined by φ respectively υ.*

Proof. Before proving a) and b), let us explain how these two parts will imply the first two statements of the proposition: For the first part, note that if $\mathcal{M}$ is étale, each $\mathcal{M}_i$ is étale and, since k is algebraically closed, $\mathcal{M}_i$ are constant $\mathcal{O}$-module schemes. In a) we show that in fact $\mathcal{M}_i$ is isomorphic to the constant $\mathcal{O}$-module scheme $(\frac{\mathcal{O}}{\pi^i})^h$ and $\bigwedge^r \mathcal{M}_i$ is isomorphic to the constant $\mathcal{O}$-module scheme $\bigwedge^r_{\frac{\mathcal{O}}{\pi^i}} (\frac{\mathcal{O}}{\pi^i})^h$. The Dieudonné module of $\mathcal{M}_i$ is therefore isomorphic to $W \otimes_{\mathbb{Z}_p} (\frac{\mathcal{O}}{\pi^i})^h \cong (W \otimes_{\mathbb{Z}_p} \frac{\mathcal{O}}{\pi^i})^h \cong (W \otimes_{\mathbb{Z}_p} \frac{\mathcal{O}}{\pi^i})^h$ and the Dieudonné module of $\bigwedge^r \mathcal{M}_i$ is isomorphic to $W \otimes_{\mathbb{Z}} \bigwedge^r_{\frac{\mathcal{O}}{\pi^i}} (\frac{\mathcal{O}}{\pi^i})^h \cong \bigwedge^r_{W\otimes_{\mathbb{Z}_p} \frac{\mathcal{O}}{\pi^i}} (W \otimes_{\mathbb{Z}_p} \frac{\mathcal{O}}{\pi^i})^h \cong \bigwedge^r_{W\otimes_{\mathbb{Z}_p} \frac{\mathcal{O}}{\pi^i}} D_i$. If $\mathcal{M}$ is connected, b) is exactly what we need to show. In the general case, write $\mathcal{M}_i$ as the direct sum, $\mathcal{M}_i^{\text{ét}} \oplus \mathcal{M}_i^{\circ}$, of its étale and connected parts (cf. Remark B.4). Using the universal properties of exterior power, tensor product and direct sum, we obtain a canonical isomorphism:

$$\bigwedge^r \mathcal{M}_i \cong \bigoplus_{j=0}^{r} (\bigwedge^j \mathcal{M}_i^{\text{ét}} \otimes \bigwedge^{r-j} \mathcal{M}_i^{\circ}).$$

Applying the covariant Dieudonné functor on the both sides of this isomorphism, and using the fact that the Dieudonné functor preserves direct sums, we get the following isomorphism (in the following isomorphisms, we omit the subscript $W \otimes_{\mathbb{Z}_p} \mathcal{O}$ from the exterior powers and tensor product in order to avoid heavy notations):

$$D_*(\bigwedge^r \mathcal{M}_i) \cong \bigoplus_{j=0}^{r} D_*(\bigwedge^j \mathcal{M}_i^{\text{ét}} \otimes \bigwedge^{r-j} \mathcal{M}_i^{\circ}).$$

Now, applying Lemma 2.38 to $\mathcal{M}_i^{\circ}$ and the étale $\mathcal{O}$-module scheme $\mathcal{M}_i^{\text{ét}}$, we can interchange the Dieudonné functor with the tensor product and we obtain:

$$D_*(\bigwedge^r \mathcal{M}_i) \cong \bigoplus_{j=0}^{r} (D_*(\bigwedge^j \mathcal{M}_i^{\text{ét}}) \otimes D_*(\bigwedge^{r-j} \mathcal{M}_i^{\circ})).$$

Finally, using parts a) and b) of the proposition, we get the following isomorphism

$$D_*(\bigwedge^r \mathcal{M}_i) \cong \bigoplus_{j=0}^{r} (\bigwedge^j D_i^{\text{ét}} \otimes \bigwedge^{r-j} D_i^{\circ})$$

where $D_i^{\text{ét}}$ and D_i° denote respectively the Dieudonné module of $\mathcal{M}_i^{\text{ét}}$ and $\mathcal{M}_i^{\circ}$. But the right hand side of the isomorphism is isomorphic to $\bigwedge^r (D_i^{\text{ét}} \oplus D_i^{\circ})$ which is itself isomorphic to $\bigwedge^r D_i$, again since the Dieudonné functor commutes with direct sums. Hence, the canonical isomorphism $D_*(\bigwedge^r \mathcal{M}_i) \cong \bigwedge^r D_i$. For the statement about the order, using the fact that the Dieudonné module of $\bigwedge^r \mathcal{M}_i$ is isomorphic to $\bigwedge^r_{W\otimes_{\mathbb{Z}_p} \frac{\mathcal{O}}{\pi^i}} D_i$ and recalling from Remark 3.13, that D_i is a free $W \otimes_{\mathbb{Z}_p} \dfrac{\mathcal{O}}{\pi^i}$-module of rank h, we deduce that

$\bigwedge\limits^r_{W\otimes_{\mathbb{Z}_p}\frac{\mathcal{O}}{\pi^i}} D_i$ is a free $W \otimes_{\mathbb{Z}_p} \dfrac{\mathcal{O}}{\pi^i}$-module of rank $\binom{h}{r}$ and that the order of $\bigwedge^r_{\frac{\mathcal{O}}{\pi^i}} \mathcal{M}_i$ is equal to $p^{\ell_W(\wedge^r D_i)} = p^{\binom{h}{r}\cdot \ell_W(W\otimes_{\mathbb{Z}_p}\frac{\mathcal{O}}{\pi^i})}$. Using Lemma 3.10 a), we have $\ell_W(W \otimes_{\mathbb{Z}_p} \frac{\mathcal{O}}{\pi^i}) = f \cdot \ell_W(W \otimes_A \frac{\mathcal{O}}{\pi^i})$. Now recall from the proof of Lemma 3.10 that in the equal characteristic case the ring A is $\mathbb{F}_q$ and in the mixed characteristic case it is the ring of integers of the maximal unramified subextension of $K/\mathbb{Q}_p$. Therefore, in the equal characteristic case we have $W \otimes_A \frac{\mathcal{O}}{\pi^i} \cong k \otimes_{\mathbb{F}_q} \frac{\mathbb{F}_q[\![\pi]\!]}{\pi^i} \cong \frac{k[\![\pi]\!]}{\pi^i}$ and in the mixed characteristic case we have $W \otimes_A \frac{\mathcal{O}}{\pi^i} \cong \frac{W}{p^i} \otimes_{\frac{A}{p^i}} \frac{\mathcal{O}}{\pi^i} \cong \frac{W}{p^i}$, since $\frac{A}{p^i} \cong \frac{\mathcal{O}}{\pi^i}$. It follows that in either case we have $\ell_W(W \otimes_A \frac{\mathcal{O}}{\pi^i}) = i$. Hence the order of $\bigwedge^r_{\frac{\mathcal{O}}{\pi^i}} \mathcal{M}_i$ is equal to $p^{fi\binom{h}{r}} = q^{i\binom{h}{r}}$. Now we prove parts a) and b).

a) Since k is algebraically closed, the finite group schemes $\mathcal{M}_i$ are constant and by abuse of notation, we will denote by $\mathcal{M}_i$ the group of k-rational points of $\mathcal{M}_i$. Again, since k is algebraically closed, we have exact sequences of $\mathcal{O}$-modules

$$(*) \qquad 0 \to \mathcal{M}_i \to \mathcal{M}_{i+1} \xrightarrow{\pi^i} \mathcal{M}_1 \to 0$$

for all natural numbers n. For $i = 1$, we have that $\mathcal{M}_i$ is an $\mathbb{F}_q$-vector space of dimension h and so it is isomorphic to $(\mathbb{F}_q)^h \cong (\frac{\mathcal{O}}{\pi})^h$. By induction on n, using exact sequence $(*)$ and straightforward calculations, one shows that $\mathcal{M}_n \cong (\frac{\mathcal{O}}{\pi^i})^h$. Thus, $\bigwedge^r_{\mathcal{O}} \mathcal{M}_i \cong \bigwedge^r_{\mathcal{O}} \underline{(\frac{\mathcal{O}}{\pi^i})^h}$ (the underline here is to emphasize that we are dealing with a constant group scheme). By Proposition 2.9 c), we know that $\bigwedge^r_{\mathcal{O}} \underline{(\frac{\mathcal{O}}{\pi^i})^h} \cong \bigwedge^r_{\frac{\mathcal{O}}{\pi^i}} \underline{(\frac{\mathcal{O}}{\pi^i})^h}$. Now the universal property of exterior powers implies that $\bigwedge^r_{\frac{\mathcal{O}}{\pi^i}} \underline{(\frac{\mathcal{O}}{\pi^i})^h} \cong \underline{\bigwedge^r_{\frac{\mathcal{O}}{\pi^i}} (\frac{\mathcal{O}}{\pi^i})^h}$ which is isomorphic to $\underline{(\frac{\mathcal{O}}{\pi^i})^{\binom{h}{r}}}$. This finishes the proof of a).

b) We know from Lemma 3.18 b) that φ and υ are commuting morphisms making the F-diagram associated to φ and the V-diagram associated to υ commute. By Remark 3.13 2), every element of $\bigwedge^r D_i$ is a linear combination of elements of the form $x_1 \wedge V x_2 \wedge \cdots \wedge V x_r$. We can therefore apply Lemma 2.36 and conclude that the covariant Dieudonné module of $\bigwedge^r_{\frac{\mathcal{O}}{\pi^i}} \mathcal{M}_i$ is isomorphic to $\bigwedge^r D_i$ with the actions of F and respectively V through the actions of φ and respectively υ.

$\square$

Remark 3.20. 1) In the proof of a) we have shown that if $\mathcal{M}$ is étale, then $\mathcal{M}_i \cong (\frac{\mathcal{O}}{\pi^i})^h$ and the injections $\iota : \mathcal{M}_i \hookrightarrow \mathcal{M}_{i+1}$ correspond to the canonical injections $(\frac{\mathcal{O}}{\pi^i})^h \hookrightarrow (\frac{\mathcal{O}}{\pi^{i+1}})^h$ given by multiplication by π. It follows that as a constant formal $\mathcal{O}$-module scheme, $\mathcal{M}$ is isomorphic to $(K/\mathcal{O})^h$.

2) Note that in the proof of the proposition, we didn't assume $r \le h$. If $r > h$, then $\bigwedge^r_{\mathcal{O}} \mathcal{M}_i = 0$ and therefore it has order $1 = q^{\binom{h}{r}}$. $\Diamond$

Corollary 3.21. *The Dieudonné module of $\bigwedge^r \mathcal{M}_i$ is canonically isomorphic to $\bigwedge^r D_i$ and the order of $\bigwedge^r \mathcal{M}_i$ is equal to $q^{i\binom{h}{r}}$.*

Proof. Fix an algebraic closure $\bar{k}$ of k. In order to simplify the notations, fix an i and set $M := \mathcal{M}_i$. By Proposition 2.12 we know that the canonical homomorphism $\bigwedge^r(M_{\bar{k}}) \to (\bigwedge^r M)_{\bar{k}}$ is an isomorphism. We then obtain the following series of isomorphisms:

$$D_*(\bigwedge^r M) \otimes_{W(k)} W(\bar{k}) \cong D_*((\bigwedge^r M)_{\bar{k}}) \cong D_*(\bigwedge^r (M_{\bar{k}})) \cong \bigwedge^r D_*(M_{\bar{k}})$$

$$\cong \bigwedge^r (D_*(M) \otimes_{W(k)} W(\bar{k})) \cong \bigwedge^r D_*(M) \otimes_{W(k)} W(\bar{k})$$

where the first and fourth isomorphisms are given by the base change property of the Dieudonné functor and the third property is given by the previous proposition, and finally, the last isomorphism is the base change property of the exterior powers in the category of modules. It follows from the construction of these isomorphisms that the resulting isomorphism

$$\bigwedge^r D_*(M) \otimes_{W(k)} W(\bar{k}) \cong D_*(\bigwedge^r M) \otimes_{W(k)} W(\bar{k})$$

is the extension of scalars of the canonical homomorphism $\vartheta : \bigwedge^r D_*(M) \to D_*(\bigwedge^r M) \cong T_{\mathrm{alt}}(M^r)$ (cf. the proof of Lemma 2.30 for the definition of this homomorphism). As the ring homomorphism $W(k) \to W(\bar{k})$ is faithfully flat, and $\vartheta \otimes_{W(k)} \mathrm{Id}$ is an isomorphism, it follows that ϑ is an isomorphism as well. The free $W(k) \otimes_{\mathbb{Z}} \mathcal{O}/\pi^i$-module $\bigwedge^r D_*(M)$ has rank $\binom{h}{r}$ and therefore, the order of $\bigwedge^r M$ is equal to $q^{i\binom{h}{r}}$. □

Remark 3.22. 1) If the ground field k is not perfect, we still have that the order of $\bigwedge^r \mathcal{M}_i$ is equal to $q^{\binom{h}{r}}$. This follows from Proposition 3.38 and the fact that the order of a group scheme is invariant under field extensions. So, we may assume that k is algebraically closed and apply Corollary 3.21.

2) It follows from Corollary 3.21 that the universal alternating morphism $\mathcal{M}_i^r \to \bigwedge^r \mathcal{M}_i$ and the universal alternating morphism $D_*(\mathcal{M}_i)^r \to \bigwedge^r D_*(\mathcal{M}_i) \cong D_*(\bigwedge^r \mathcal{M}_i)$ correspond to each other under the isomorphism $\mathrm{Alt}^{\mathcal{O}}(D_*(\mathcal{M}_i)^r, D_*(\bigwedge^r \mathcal{M}_i)) \cong \mathrm{Alt}^{\mathcal{O}}(\mathcal{M}_i^r, \bigwedge^r \mathcal{M}_i)$ explained in Remark 1.24. ◊

Notations 3.23. Unless otherwise specified, k is a field of characteristic $p > 2$ (not necessarily perfect).

Construction 3.24. By Proposition 2.25 we have the following exact sequence:

$$\bigwedge^r \mathcal{M}_{n+m} \xrightarrow{\pi^m} \bigwedge^r \mathcal{M}_{n+m} \to \bigwedge^r \mathcal{M}_m \to 0$$

and since the composition $\pi^n \circ \pi^m$ is zero on $\bigwedge^r \mathcal{M}_{n+m}$, the morphism $\bigwedge^r \mathcal{M}_{n+m} \xrightarrow{\pi^n} \bigwedge^r \mathcal{M}_{n+m}$, factors through the epimorphism $\bigwedge^r \mathcal{M}_{n+m} \twoheadrightarrow \bigwedge^r \mathcal{M}_m$:

$$\begin{tikzcd} \bigwedge^r \mathcal{M}_{n+m} \arrow[rr, "\pi^n"] \arrow[dr] & & \bigwedge^r \mathcal{M}_{n+m} \\ & \bigwedge^r \mathcal{M}_m . \arrow[ur, "\eta"'] & \end{tikzcd} \tag{3.25}$$

▼

Lemma 3.26. *The sequence* $0 \to \bigwedge^r \mathcal{M}_m \xrightarrow{\eta} \bigwedge^r \mathcal{M}_{n+m} \to \bigwedge^r \mathcal{M}_n \to 0$ *is exact.*

Proof. By construction of η and the fact that $\bigwedge^r \mathcal{M}_{n+m} \twoheadrightarrow \bigwedge^r \mathcal{M}_m$ is an epimorphism, we know that the sequence $\bigwedge^r \mathcal{M}_m \xrightarrow{\eta} \bigwedge^r \mathcal{M}_{n+m} \to \bigwedge^r \mathcal{M}_n \to 0$ is exact. By previous remark, we have

$$\mid \bigwedge^r \mathcal{M}_m \mid \cdot \mid \bigwedge^r \mathcal{M}_n \mid = q^{m\binom{h}{r}} q^{n\binom{h}{r}} = q^{(n+m)\binom{h}{r}} = \mid \bigwedge^r \mathcal{M}_{m+n} \mid .$$

The fact that the order of finite group schemes is multiplicative in short exact sequences implies that η is indeed a monomorphism. □

Remark 3.27. Switching m and n in diagram (3.25), we obtain the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \bigwedge^r \mathcal{M}_m \arrow[r, "\eta"] & \bigwedge^r \mathcal{M}_{n+m} \arrow[r] \arrow[d, "\pi^m"] & \bigwedge^r \mathcal{M}_n \arrow[r] \arrow[dl, hook, "\eta"'] & 0 \\ & & \bigwedge^r \mathcal{M}_{n+m} & & \end{tikzcd}$$

from which we obtain the following exact sequence:

$$0 \to \bigwedge^r \mathcal{M}_m \xrightarrow{\eta} \bigwedge^r \mathcal{M}_{n+m} \xrightarrow{\pi^m} \bigwedge^r \mathcal{M}_{n+m}. \tag{3.28}$$

If we regard $\bigwedge^r \mathcal{M}_n$ as a submodule scheme of $\bigwedge^r \mathcal{M}_{n+m}$, using the monomorphism $\eta : \bigwedge^r \mathcal{M}_n \hookrightarrow \bigwedge^r \mathcal{M}_{n+m}$, we may as well denote the epimorphism $\bigwedge^r \mathcal{M}_{n+m} \twoheadrightarrow \bigwedge^r \mathcal{M}_n$ by π^m, i.e., multiplication by π^m and rewrite the first exact sequence of the remark as:

$$0 \to \bigwedge^r \mathcal{M}_m \xrightarrow{\eta} \bigwedge^r \mathcal{M}_{n+m} \xrightarrow{\pi^m} \bigwedge^r \mathcal{M}_n \to 0. \tag{3.29}$$

$\diamondsuit$

Proposition 3.30. *Let $\mathcal{M}$ be a one dimensional π-divisible module over k, of height h. Denote by Λ^r the following inductive system*

$$\bigwedge^r \mathcal{M}_1 \overset{\eta}{\hookrightarrow} \bigwedge^r \mathcal{M}_2 \overset{\eta}{\hookrightarrow} \bigwedge^r \mathcal{M}_3 \hookrightarrow \cdots$$

viewed as an ind-object in the category of finite group schemes over k. Then Λ^r is a π-divisible module over k, of height $\binom{h}{r}$, and together with the system of universal alternating morphisms $\lambda_n : \mathcal{M}_n^r \to \bigwedge^r \mathcal{M}_n$, it is the r^{th}-exterior power of $\mathcal{M}$, i.e., we have $\Lambda^r \cong \bigwedge^r \mathcal{M}$.

Proof. Fix a natural number m. By Lemma 2.36, we know that $\bigwedge^r \mathcal{M}_m$ exists as an $\mathcal{O}$-module scheme, and its order is equal to $q^{m\binom{h}{r}}$ by Remark 3.22 1). Setting $n = 1$ in the exact sequence (3.28), we obtain the following exact sequence:

$$0 \to \bigwedge^r \mathcal{M}_m \xrightarrow{\eta} \bigwedge^r \mathcal{M}_{m+1} \xrightarrow{\pi^m} \bigwedge^r \mathcal{M}_{m+1}.$$

By Remark B.3, these two properties (the equality $|\bigwedge^r \mathcal{M}_m| = q^{m\binom{h}{r}}$ and the exact sequence) imply that the direct limit $\Lambda^r = \varinjlim(\bigwedge^r \mathcal{M}_n, \eta)$ is a π-divisible module and we have $\operatorname{Ker}(\pi^m : \Lambda^r \to \Lambda^r) \cong \bigwedge^r \mathcal{M}_m$. The height of Λ^r is equal to $\log_q |\bigwedge^r \mathcal{M}_1| = \binom{h}{r}$, as claimed. The projections $\bigwedge^r \mathcal{M}_{n+1} \twoheadrightarrow \bigwedge^r \mathcal{M}_n$ are induced by the universal alternating morphisms $\lambda_{n+1} : \mathcal{M}_{n+1}^r \to \bigwedge^r \mathcal{M}_{n+1}$ and the alternating morphisms $\mathcal{M}_{n+1}^r \overset{(\pi.)^r}{\twoheadrightarrow} \mathcal{M}_n^r \xrightarrow{\lambda_n} \bigwedge^r \mathcal{M}_n$. Thus, the system of alternating morphisms $\lambda := (\lambda_n)_{n\geq 1}$ is an element of the $\mathcal{O}$-module $\operatorname{Alt}_k^{\mathcal{O}}(\mathcal{M}^r, \Lambda^r)$ with the following universal property: for every π-divisible module $\mathcal{N}$ over k the $\mathcal{O}$-linear homomorphism

$$\lambda^* : \operatorname{Hom}_k(\Lambda^r, \mathcal{N}) \longrightarrow \operatorname{Alt}_k^{\mathcal{O}}(\mathcal{M}^r, \mathcal{N})$$

induced by λ is an isomorphism. This proves the last part of the proposition. $\square$

Proposition 3.31. *Assume k is perfect and let $\mathcal{M}$ be a one dimensional π-divisible module over k. Then the covariant Dieudonné module of $\bigwedge^r \mathcal{M}$ is isomorphic to $\bigwedge^r D$.*

Proof. By definition, the covariant Dieudonné module of $\bigwedge^r \mathcal{M}$ is the inverse limit of the system $D_*(\bigwedge^r \mathcal{M}_{i+1}) \overset{\zeta}{\to} D_*(\bigwedge^r \mathcal{M}_i)$. By Corollary 3.21, we know that $D_*(\bigwedge^r \mathcal{M}_i) = \bigwedge^r D_i$. So, we only need to show that

$$\bigwedge^r_{W\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O}} \varprojlim D_i = \varprojlim \bigwedge^r_{W\otimes_{\mathbb{Z}_p}\frac{\mathcal{O}}{\pi^i}} D_i.$$

Writing D for the inverse limit $\varprojlim D_i$ as before, we have natural and compatible morphisms $\bigwedge^r D \to \bigwedge^r D_i$ (for all $i \geq 1$) that we denoted by $\bigwedge^r \zeta$. These were induced by the natural morphisms $\zeta : D \to D_i$. So, we obtain a morphism $\psi : \bigwedge^r D \to \varprojlim \bigwedge^r D_i$. Explicitly, ψ sends an element $d_1 \wedge \cdots \wedge d_r \in \bigwedge^r D$ to the element $(\zeta_i(d_1) \wedge \cdots \wedge \zeta_i(d_r)) \in \varprojlim \bigwedge^r D_i$ (here we write down the index i for the morphism ζ to emphasize that the element $\zeta_i(d_1) \wedge \cdots \wedge \zeta_i(d_r)$ is the i^{th} component). Now, the result is a formal consequence of the fact that D is a finite free $W\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O}$-module (cf. Lemma 3.12), that $D_i = D/\pi^i D$ is a free $W \otimes_{\mathbb{Z}_p} \mathcal{O}/\pi^i$-module and that $W\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O}$ is complete. $\square$

Lemma 3.32. *Let $\mathcal{B}$ be a discrete valuation ring with valuation $v : \mathcal{B} \to \mathbb{Z} \cup \{\infty\}$ and endowed with an automorphism $\omega : \mathcal{B} \to \mathcal{B}$. Let $\chi_i : M_i \to N_i$ $(i = 1, \ldots, n)$ be ω-linear morphisms between finite free $\mathcal{B}$-modules of the same rank. Then, we have*

$$\ell_{\mathcal{B}}(\operatorname{Coker}(\chi)) = v(det(\chi_1) \cdot det(\chi_2) \cdots det(\chi_n))$$

where $\chi : M_1 \oplus M_2 \oplus \cdots \oplus M_n \to N_1 \oplus N_2 \oplus \cdots \oplus N_n$ is the direct sum of χ_i.

Proof. Since M_i are free $\mathcal{B}$-modules, the automorphism ω induces an ω^{-1}-linear automorphism, ω_i, of M_i. More precisely, if we choose a basis $\{m_\alpha \mid \alpha \in \Lambda\}$ of M_i, then $\omega_i(\sum_{\alpha\in\Lambda} b_\alpha m_\alpha) = \sum_{\alpha\in\Lambda} \omega^{-1}(b_\alpha) m_\alpha$. Now, the composite $\chi_i \circ \omega_i : M_i \to N_i$ is ω-linear and we have $\mathrm{Coker}(\chi) \cong \mathrm{Coker}(\bigoplus(\chi_i \circ \omega_i))$ and $\det(\chi_i) = \det(\chi_i \circ \omega_i)$ for all $1 \leq i \leq n$. Replacing χ_i by $\chi_i \circ \omega_i$, we may assume that χ_i are $\mathcal{B}$-linear. First assume $n = 1$ and let s be the rank of M_1. By the elementary divisor theorem, there is a basis of M_1 and a basis of N_1 such that the matrix of χ_1 in this basis is diagonal (Smith normal form), with diagonal entries $a_j \in \mathcal{B}$ $(j = 1, \ldots, s)$ and we can assume $a_j \neq 0\,(\forall j)$. It follows that the determinant of χ_1 is equal to $a_1 \cdot a_2 \cdots a_s$ and that the cokernel of χ_1 is isomorphic to $\dfrac{\mathcal{B}}{(a_1)} \oplus \dfrac{\mathcal{B}}{(a_2)} \oplus \cdots \oplus \dfrac{\mathcal{B}}{(a_s)}$. The result follows at once (note that $\ell_{\mathcal{B}}(\dfrac{\mathcal{B}}{(a_j)}) = v(a_j)$). In the general case, we have that $\mathrm{Coker}(\chi) = \mathrm{Coker}(\chi_1) \oplus \mathrm{Coker}(\chi_2) \oplus \cdots \oplus \mathrm{Coker}(\chi_n)$ and the result follows from the case $n = 1$. $\square$

Lemma 3.33. *Let $\mathcal{N}$ be a finite dimensional π-divisible module over a perfect field k. Then we have*

$$\dim \mathcal{N} = \ell_W(\mathrm{Coker}(V_{D_*(\mathcal{N})})) = \ell_{W\widehat{\otimes}_A \mathcal{O}}(\mathrm{Coker}(V_{D_*(\mathcal{N})}))$$

where A is the ring defined in Lemma 3.10.

Proof. As $\mathcal{N}$ is finite dimensional, the dimension of $\mathcal{N}$ is equal to the dimension of $\mathcal{N}_i$ for all $i \gg 0$, where as usual $\mathcal{N}_i$ denotes the kernel of π^i. We have a canonical isomorphism $\mathrm{Coker}(V_{D_*(\mathcal{N}_i)}) \cong \mathcal{L}ie(\mathcal{N}_i)$, which implies that

$$\dim \mathcal{N}_i = \dim_k \mathrm{Coker}(V_{D_*(\mathcal{N}_i)}) = \ell_W(\mathrm{Coker}(V_{D_*(\mathcal{N}_i)})). \tag{3.34}$$

Since $V : D_*(\mathcal{N}_i) \to D_*(\mathcal{N}_i)$ is a morphism between finite length W-modules, we have

$$\mathrm{Coker}(V_{D_*(\mathcal{N})}) \cong \mathrm{Coker}(\varprojlim_i V_{D_*(\mathcal{N}_i)}) \cong \varprojlim_i \mathrm{Coker}(V_{D_*(\mathcal{N}_i)}). \tag{3.35}$$

The projections $D_*(\mathcal{N}_{i+1}) \twoheadrightarrow D_*(\mathcal{N}_i)$ induce surjections between the cokernels of $V_{D_*(\mathcal{N}_{i+1})}$ and $V_{D_*(\mathcal{N}_i)}$, and since for $i \gg 0$, cokernels of $V_{D_*(\mathcal{N}_{i+1})}$ and $V_{D_*(\mathcal{N}_i)}$ have the same dimension, the induced surjections are isomorphisms for $i \gg 0$. Equations (3.34) and (3.35) imply that $\dim \mathcal{N} = \ell_W(\mathrm{Coker}(V_{D_*(\mathcal{N})}))$. For the second equality of the lemma, it is enough to show that for $i \gg 0$, we have $\ell_W(\mathrm{Coker}(V_{D_*(\mathcal{N}_i)})) = \ell_{W\widehat{\otimes}_A \mathcal{O}}(\mathrm{Coker}(V_{D_*(\mathcal{N}_i)}))$ (note that $\mathrm{Coker}(V_{D_*(\mathcal{N})}) \cong \mathrm{Coker}(V_{D_*(\mathcal{N}_i)})$ for $i \gg 0$). As $D_*(\mathcal{N}_i)$ is killed by π^i, and $(W\widehat{\otimes}_A \mathcal{O})/\pi^i \cong W \otimes_A \dfrac{\mathcal{O}}{\pi^i}$, we have $\ell_{W\widehat{\otimes}_A \mathcal{O}}(\mathrm{Coker}(V_{D_*(\mathcal{N}_i)})) = \ell_{W\otimes_A \frac{\mathcal{O}}{\pi^i}}(\mathrm{Coker}(V_{D_*(\mathcal{N}_i)}))$. As we have seen before (in the proof of Proposition 3.19), we have $W \otimes_A \dfrac{\mathcal{O}}{\pi^i} \cong \dfrac{W}{p^i}$, and therefore we obtain $\ell_{W\otimes_A \frac{\mathcal{O}}{\pi^i}}(\mathrm{Coker}(V_{D_*(\mathcal{N}_i)})) = \ell_{\frac{W}{p^i}}(\mathrm{Coker}(V_{D_*(\mathcal{N}_i)})) = \ell_W(\mathrm{Coker}(V_{D_*(\mathcal{N}_i)}))$. These equalities together with the last one imply the desired equality. $\square$

Theorem 3.36. *Let $\mathcal{M}$ be a π-divisible module of height h and dimension 1 over a field k of characteristic $p > 2$. Then the r^{th}-exterior power of $\mathcal{M}$ in the category of π-divisible modules over k exists, and has height $\binom{h}{r}$ and dimension $\binom{h-1}{r-1}$. Moreover, for every n, we have $(\bigwedge_{\mathcal{O}}^r \mathcal{M})_n \cong \bigwedge_{\mathcal{O}}^r(\mathcal{M}_n)$.*

Proof. We already know by Proposition 3.30 that $\bigwedge^r \mathcal{M}$ is a π-divisible module over k, of height $\binom{h}{r}$. So, we should only calculate the dimension of $\bigwedge^r \mathcal{M}$. Since the dimension is invariant under the base change, by Proposition 3.38 we may assume that k is algebraically closed. Using Lemma 3.33, we know that the dimension of $\bigwedge^r \mathcal{M}$ is equal to the length of the cokernel of the Verschiebung of the covariant Dieudonné module of $\bigwedge^r \mathcal{M}$, which is by Proposition 3.31 isomorphic to $\bigwedge^r D$, and thus, we have to calculate $\ell_{W\widehat{\otimes}_A \mathcal{O}}(\mathrm{Coker}(\bigwedge^r V))$. By Lemma 3.10, we have $D = M_0 \times M_1 \times \cdots \times M_{f-1}$, where M_i are $W\widehat{\otimes}_A \mathcal{O}$-modules and the Verschiebung induces ${}^{\sigma^{-1}} \otimes \mathrm{Id}$-linear morphisms $V : M_i \to M_{i+1}$ for all $i \in \frac{\mathbb{Z}}{f\mathbb{Z}}$, which we denote by V_i. Again, by Proposition 3.31, the Verschiebung of $\bigwedge^r D$ is the morphism $\Upsilon = \bigwedge^r V$. It is now straightforward to check that

$$\bigwedge^r D = \bigwedge^r M_0 \times \bigwedge^r M_1 \times \cdots \times \bigwedge^r M_{f-1}$$

and that the morphism $(\bigwedge^r V)_i : \bigwedge^r M_i \to \bigwedge^r M_{i+1}$ induced by $\bigwedge^r V$ is equal to $\bigwedge^r(V_i)$. Now, recalling from Lemma 3.10 that $W\widehat{\otimes}_A\mathcal{O}$ is a discrete valuation ring, and denoting its valuation by v, we have by Lemma 3.32 (here we are using the fact that D and $\bigwedge^r D$ are free $W\widehat{\otimes}_A\mathcal{O}$-modules; cf. Lemma 3.12):

$$\ell_{W\widehat{\otimes}_A\mathcal{O}}(\mathrm{Coker}(\bigwedge^r V)) = v(\det(\bigwedge^r V_1)\cdots\det(\bigwedge^r V_f)) = v(\det(V_1)^{\binom{h-1}{r-1}}\cdots\det(V_f)^{\binom{h-1}{r-1}}) =$$

$$\binom{h-1}{r-1}\cdot v(\det(V_1)\cdots\det(V_f)) = \binom{h-1}{r-1}\cdot \ell_{W\widehat{\otimes}_A\mathcal{O}}(\mathrm{Coker}(V)) = \binom{h-1}{r-1}\cdot \dim\mathcal{M} = \binom{h-1}{r-1}$$

where the second equality is true because V_i are semi-linear morphisms between finite free modules and we have the relation between the determinant of V_i and that of $\bigwedge^r V_i$ which we mentioned in the proof of Lemma 3.14, and the penultimate equality follows from Lemma 3.33. The last statement of the theorem follows from the construction of $\bigwedge^r\mathcal{M}$ (cf. Proposition 3.30). ☐

3.4 The main theorem: over arbitrary fields

We need the following lemma for proving that the base change homomorphism is an isomorphism for every filed extension:

Lemma 3.37. *Let $\psi : G \to H$ be a homomorphism of affine group schemes of finite type over* $\mathbf{Spec}(k)$*, where k is a field. Assume that for every finite group scheme I over k, the induced homomorphism of groups*

$$\psi_*(I) : \mathrm{Hom}(I, G) \to \mathrm{Hom}(I, H)$$

is an isomorphism and also the induced homomorphism on $\bar{k}$-valued points, $\psi(\bar{k}) : G(\bar{k}) \to H(\bar{k})$, is an isomorphism. Then ψ is an isomorphism.

Proof. We show at first that ψ is a monomorphism. Denote by K the kernel of ψ. Since G is of finite type over k, its closed subgroup K is also of finite type over k. The sequence

$$0 \longrightarrow K(\bar{k}) \longrightarrow G(\bar{k}) \xrightarrow{\psi(\bar{k})} H(\bar{k})$$

is exact, but by assumption, $\psi(\bar{k})$ is injective and therefore, $K(\bar{k}) = 0$. It follows that K is a finite group scheme over k. By assumption the homomorphism $\psi_*(K) : \mathrm{Hom}(K, G) \to \mathrm{Hom}(K, H)$ is injective. It implies that the inclusion $K \hookrightarrow G$ is the zero homomorphism and thus $K = 0$.

In order to show that ψ is an epimorphism, we consider the problem over fields of positive characteristic and characteristic zero separately. First, the case when k has positive characteristic p. Assume at first that H is connected. Let $H[F^n]$ denote the kernel of the homomorphism $F^n_H : H \to H^{(p^n)}$. As H is a scheme of finite type over k, the subgroup schemes $H[F^n]$ are finite over k, for every n. It follows from the assumption that the inclusion $H[F^n] \hookrightarrow H$ factors through the inclusion $G \hookrightarrow H$. Denote by I_H the augmentation ideal of H and by J the ideal in $\mathcal{O}(H)$ (the coordinate ring of H) defining G, i.e., we have $\mathcal{O}(G) \cong \mathcal{O}(H)/J$. Since G contains the kernel of all powers of the Frobenius morphism of H, we have $J \subseteq \bigcap_{n=1}^{\infty} I_H^n$. But this intersection is trivial, because H is connected. Hence $\mathcal{O}(G) \cong \mathcal{O}(H)$ and $G \cong H$. In the general case, denote by H^0 the connected component of H, containing the zero section, and by G_0 the intersection $G \cap H^0$. The hypotheses of the proposition hold for the induced homomorphism $\psi_{|G_0} : G_0 \to H^0$ and since H^0 is connected, by the above arguments, we have $G_0 = H^0$. This shows that G contains H^0. As H is of finite type over k, it has finitely many connected components. Thus, the quotient H/H^0 is a finite étale group scheme. This finite quotient surjects onto the quotient H/G, which implies that H/G is a finite étale group scheme over k. Consider the following short exact sequence:

$$0 \to G \xrightarrow{\psi} H \longrightarrow H/G \to 0.$$

Taking the $\bar{k}$-valued points, we obtain the following short exact sequence:

$$0 \to G(\bar{k}) \xrightarrow{\psi(\bar{k})} H(\bar{k}) \longrightarrow H/G(\bar{k}) \to 0.$$

Since by assumption $\psi(\bar{k})$ is an isomorphism, we have that $(H/G)(\bar{k})$ is trivial. As H/G is étale and is trivial on $\bar{k}$-valued points, the group scheme H/G is trivial as well. Hence ψ is an isomorphism.

Now assume that k is a field of characteristic zero. Since $\psi(\bar{k})$ is surjective, ψ is a dominant morphism. It follows that the kernel of the ring homomorphism $\psi^\sharp : \mathcal{O}(H) \to \mathcal{O}(G)$ between the Hopf algebras of H and G is nilpotent. Since k is of characteristic zero, H is reduced (Cartier's theorem) and therefore the kernel of $\psi^\sharp$ is zero, which means that $\psi : G \to H$ is an epimorphism. $\square$

Proposition 3.38. *The base change homomorphism $f : \bigwedge^r(\mathcal{M}_{n,\ell}) \to (\bigwedge^r \mathcal{M}_n)_\ell$ is an isomorphism for every field extension ℓ/k.*

Proof. First assume that ℓ is an algebraically closed field and consider the homomorphism

$$\lambda^* : \underline{\mathrm{Hom}}_\ell(\bigwedge^r(\mathcal{M}_{n,\ell}), \mathbb{G}_m) \to \widetilde{\underline{\mathrm{Alt}}}_\ell^{\mathcal{O}}(\mathcal{M}^r_{n,\ell}, \mathbb{G}_m)$$

obtained by universal property of $\bigwedge^r(\mathcal{M}_{n,\ell})$ and sheafification. This homomorphism induces an isomorphism on the ℓ-valued points (this is the universal property of $\bigwedge^r(\mathcal{M}_{n,\ell})$). For every finite group scheme I over ℓ, we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_\ell(I, \underline{\mathrm{Hom}}_\ell(\bigwedge^r(\mathcal{M}_{n,\ell}), \mathbb{G}_m)) & \xrightarrow{\mathrm{Hom}(I,\lambda^*)} & \mathrm{Hom}_\ell(I, \widetilde{\underline{\mathrm{Alt}}}_\ell^{\mathcal{O}}(\mathcal{M}^r_{n,\ell}, \mathbb{G}_m)) \\
\cong \downarrow & & \downarrow \cong \ (1.7) \\
\mathrm{Hom}_\ell(\bigwedge^r(\mathcal{M}_{n,\ell}), \underline{\mathrm{Hom}}_\ell(I, \mathbb{G}_m)) & \longrightarrow & \widetilde{\mathrm{Alt}}_\ell^{\mathcal{O}}(\mathcal{M}^r_{n,\ell}, \underline{\mathrm{Hom}}_\ell(I, \mathbb{G}_m)).
\end{array}$$

Since I is a finite group scheme over ℓ, $\underline{\mathrm{Hom}}_\ell(I, \mathbb{G}_m)$ is a (finite) group scheme over ℓ and so, by the universal property of $\bigwedge^r(\mathcal{M}_{n,\ell})$, the bottom morphism of the diagram is an isomorphism, which implies that the top morphism is an isomorphism too. We can know apply Lemma 3.37 and conclude that λ^* is an isomorphism. In particular, since by Corollary 3.21, $\bigwedge^r(\mathcal{M}_{n,\ell})$ is finite over ℓ, it follows that $\widetilde{\underline{\mathrm{Alt}}}_\ell^{\mathcal{O}}(\mathcal{M}^r_{n,\ell}, \mathbb{G}_m)$ is finite over ℓ as well. From the isomorphism $\widetilde{\underline{\mathrm{Alt}}}_k^{\mathcal{O}}(\mathcal{M}^r_n, \mathbb{G}_m)_\ell \cong \widetilde{\underline{\mathrm{Alt}}}_\ell^{\mathcal{O}}(\mathcal{M}^r_{n,\ell}, \mathbb{G}_m)$ and the finiteness of $\widetilde{\underline{\mathrm{Alt}}}_\ell^{\mathcal{O}}(\mathcal{M}^r_{n,\ell}, \mathbb{G}_m)$ over ℓ we deduce that $\widetilde{\underline{\mathrm{Alt}}}_k^{\mathcal{O}}(\mathcal{M}^r_n, \mathbb{G}_m)$ is finite over k. Now, let ℓ be any extension of k. By Theorem 2.5, $\bigwedge^r \mathcal{M}_n$ is isomorphic to $\widetilde{\underline{\mathrm{Alt}}}_k^{\mathcal{O}}(\mathcal{M}^r_n, \mathbb{G}_m)^*$ which is equal to the Cartier dual of $\widetilde{\underline{\mathrm{Alt}}}_k^{\mathcal{O}}(\mathcal{M}^r_n, \mathbb{G}_m)$ (since it is finite). So, we have shown that for every field k, we have a canonical isomorphism $\bigwedge^r \mathcal{M}_n \cong \widetilde{\underline{\mathrm{Alt}}}_k^{\mathcal{O}}(\mathcal{M}^r_n, \mathbb{G}_m)$. As the Cartier duality and the construction $\widetilde{\underline{\mathrm{Alt}}}^{\mathcal{O}}$ commute with base change, we conclude that the base change homomorphism $f : \bigwedge^r(\mathcal{M}_{n,\ell}) \to (\bigwedge^r \mathcal{M}_n)_\ell$ is an isomorphism as desired. $\square$

Theorem 3.39. *Let $\mathcal{M}$ be a π-divisible module of height h over a base field k. Assume that the dimension of $\mathcal{M}$ is at most 1. Fix a positive natural number r. Then the r^{th}-exterior power of $\mathcal{M}$ in the category of π-divisible modules over k exists, and has height $\binom{h}{r}$. If the dimension of $\mathcal{M}$ is 1, then $\bigwedge^r_{\mathcal{O}} \mathcal{M}$ has dimension $\binom{h-1}{r-1}$, otherwise, it has dimension zero. Moreover, for every positive natural number n, we have $(\bigwedge^r_{\mathcal{O}} \mathcal{M})_n \cong \bigwedge^r_{\mathcal{O}}(\mathcal{M}_n)$. Furthermore, for any field extension ℓ/k, the base change morphism $\bigwedge^r_{\mathcal{O}}(\mathcal{M}_\ell) \to (\bigwedge^r_{\mathcal{O}} \mathcal{M})_\ell$ is an isomorphism.*

Proof. If the characteristic of k is different from p or the dimension of $\mathcal{M}$ is zero, then $\mathcal{M}$ is étale and the statements of the theorem follow from Proposition 3.3 and Remark 3.4. So, we can assume that the characteristic of k is p and the dimension of $\mathcal{M}$ is 1. The statements of the theorem now follow from Theorem 3.36 and Proposition 3.38. $\square$

A Two Results of R. Pink

In this section, we present two results from [6] (in the form) that we need. For the sake of completeness and because this paper is not yet publicly available, we also include their proofs or an sketch thereof. We emphasize on the fact that these results and their proofs are entirely due to R. Pink.

Proposition A.1. *Let $G_1, \dots, G_r$ be profinite group schemes over a field k and H an affine group scheme of finite type over k. Then for any multilinear morphism $\varphi : G_1 \times \cdots \times G_r \to H$, there is a factorization*

$$\begin{tikzcd}
G_1 \times \cdots \times G_r \arrow[r] \arrow[d] & H \\
G'_1 \times \cdots \times G'_r \arrow[r] & H' \arrow[u, hook]
\end{tikzcd}$$

where $G_i \twoheadrightarrow G'_i$ are quotients with G'_i finite and H' is a finite subgroup of H.

Proof. Write $G_i = \mathbf{Spec}(A_i)$ and $H = \mathbf{Spec}(B)$ and let $\varphi^{\#} : B \to A_r \otimes \cdots \otimes A_r$ be the k-algebra map corresponding to φ. Each A_i is a direct limit of finite k-algebras, and therefore every finitely generated sub-k-algebra of A_i is again a finite k-algebra. As B is finitely generated over k, the sub-k-algebra $\varphi^{\#}(B)$ is contained in $A'_r \otimes \cdots \otimes A'_r$ for some finitely generated sub-k-algebras A'_i of A_i. After enlarging A'_i if necessary, we may assume they are Hopf subalgebras and therefore, φ factors through $G'_1 \times \cdots \times G'_r$ where $G_i = \mathbf{Spec}(A'_i)$ are finite group schemes over k. The projection $G_1 \times \cdots \times G_r \twoheadrightarrow G'_1 \times \cdots \times G'_r$ being an epimorphism, the induced morphism $G'_1 \times \cdots \times G'_r \to H$ is multilinear as well. So, we may assume that G_i are finite over k. If all G_i are étale over k, then φ corresponds to a multilinear morphism of abelian groups $\varphi(k^s) : G_1(k^s) \times \cdots \times G_r(k^s) \to H(k^s)$, which is invariant under $\mathrm{Gal}(k^s/k)$. This multilinear map corresponds to a homomorphism of $\mathrm{Gal}(k^s/k)$-modules $\Gamma := G_1(k^s) \otimes \cdots \otimes G_r(k^s) \to H(k^s)$. This gives rise to a $\mathrm{Gal}(k^s/k)$-invariant homomorphism of group schemes $\underline{\varphi} : \underline{\Gamma}_{k^s} \to H_{k^s}$. The constant group $\underline{\Gamma}_{k^s}$ being finite, the image of $\underline{\varphi}$ is a $\mathrm{Gal}(k^s/k)$-invariant and finite subgroup $H'_{k^s} \subseteq H_{k^s}$. Thus, H'_{k^s} descends to a finite subgroup H' of H and so, φ factors through H'. If G_i are not all étale, then the characteristic of k is positive, say p. Choose n large enough so that for all i, the n-th iteration of the Frobenius $F_i^n : G_i \to G_i^{(p^n)}$ kills the identity component G_i°. Consider the following commutative diagram deduced from Proposition 1.8 *a*):

$$\begin{tikzcd}
G_1 \times \cdots \times G_r \arrow[r, "\varphi"] \arrow[d] \arrow[dd, bend right=60, "F^n_{G_1} \times \cdots \times F^n_{G_r}"'] & H \arrow[dd, "F^n_H"] \\
G_1^{\text{ét}} \times \cdots \times G_r^{\text{ét}} \arrow[d, hook, "\iota"'] & \\
G_1^{(p^n)} \times \cdots \times G_r^{(p^n)} \arrow[r, "\varphi^{(p^n)}"'] & H^{(p^n)}.
\end{tikzcd}$$

By the étale case above, the composition $\varphi^{(p^n)} \circ \iota$ factors through a finite subgroup $\tilde{H}$ of $H^{(p^n)}$. Therefore, φ factors through the preimage of $\tilde{H}$ under F_H^n, which is a finite group as F_H is a finite morphism. $\square$

Proposition A.2. *Let R be a ring and k a perfect field of positive characteristic p. For any sheaf of R-modules $\mathcal{F}$ over k, we let $\widehat{\mathbb{E}}_k \otimes R$ act on $\mathrm{Hom}(\mathbb{W}, \mathcal{F})$ as follows: $(e \otimes x) \cdot f := x \circ f \circ e^*$, where $(\_)^*$ is the usual involution of $\widehat{\mathbb{E}}_k$, being identity on $W(k)$ and interchanging F and V. Now, let G be a finite R-module scheme over k which is a quotient of $\mathbb{W}^n$ for some n. Then the homomorphism*

$$\mathrm{Hom}_R(G, \mathcal{F}) \to \mathrm{Hom}_{\widehat{\mathbb{E}}_k \otimes R}(\mathrm{Hom}(\mathbb{W}, G), \mathrm{Hom}(\mathbb{W}, \mathcal{F})), \quad \alpha \mapsto (f \mapsto \alpha \circ f)$$

is an isomorphism.

Proof. First assume that $R = \mathbb{Z}$ (this the case done in [6]). By assumption on G, there exists a presentation $\mathbb{W}^m \xrightarrow{v} \mathbb{W}^n \xrightarrow{u} G \to 0$. Let $u_i : \mathbb{W} \to G$ and $v_{ij} : \mathbb{W} \to \mathbb{W}$ be the components of u and v. Suppose that $\alpha \in \mathrm{Hom}(G, \mathcal{F})$ maps to zero. So, $\alpha \circ u_i = 0$ and so $\alpha \circ u = 0$, which implies that $\alpha = 0$, since u is an epimorphism. Now, let $\varphi : \mathrm{Hom}(\mathbb{W}, G) \to \mathrm{Hom}(\mathbb{W}, \mathcal{F})$ be $\widehat{\mathbb{E}}_k$-linear. Elements $\varphi(u_i) : \mathbb{W} \to \mathcal{F}$ give rise to an element $\alpha \in \mathrm{Hom}(\mathbb{W}^n, \mathcal{F})$. Then $\alpha \circ v$ has components $\sum_i \varphi(u_i) \circ v_{ij} = \varphi(\sum_i u_i \circ v_{ij})$, by linearity, which is zero, as $u \circ v = 0$. So, $\alpha = \beta \circ u$ for some $\beta : G \to \mathcal{F}$. We want to show that β maps to φ, proving the surjectivity. We ought to show that for all $f : \mathbb{W} \to G$, we have $\varphi(f) = \beta \circ f$. There exists an element $g : \mathbb{W} \to \mathbb{W}^n$ such that $f = u \circ g$, with components $g_i \in \widehat{\mathbb{E}}_k$. Then, we have

$$\varphi(f) = \varphi(u \circ g) = \varphi(\sum_i u_i \circ g_i) = \sum_i \varphi(u_i) \circ g_i = \alpha \circ g = \beta \circ u \circ g = \beta \circ f.$$

Now, let R be arbitrary. It is clear that the image of $\mathrm{Hom}_R(G, \mathcal{F})$ under the above isomorphism lands inside the subgroup $\mathrm{Hom}_{\widehat{\mathbb{E}}_k \otimes R}(\mathrm{Hom}(\mathbb{W}, G), \mathrm{Hom}(\mathbb{W}, \mathcal{F}))$ and therefore, it is enough to show that if the image of an element α is in this subgroup, then α is R-linear. This means that we have to show that for every $x \in R$, the two composites $x \circ \alpha$ and $\alpha \circ x$ are equal. By assumption, the image of these two maps are equal, and the injectivity implies that they are equal as well. □

B π-Divisible Modules

In this section, we define and prove some properties of π-divisible modules.

Definition B.1. Let S be a scheme and $\mathcal{M}$ an fppf sheaf of $\mathcal{O}$-modules over S. We call $\mathcal{M}$ a *π-Barsotti-Tate group* or *π-divisible module over S* if the following conditions are satisfied:

(i) $\mathcal{M}$ is π-divisible, i.e., the homomorphism $\pi : \mathcal{M} \to \mathcal{M}$ is an epimorphism.

(ii) $\mathcal{M}$ is π-torsion, i.e., the canonical homomorphism $\varinjlim_n \mathcal{M}[\pi^n] \to \mathcal{M}$ is an isomorphism.

(iii) $\mathcal{M}[\pi]$ is representable by a finite locally free $\mathcal{O}$-module scheme over S.

The order of $\mathcal{M}[\pi]$ is of the form q^h, where $h : S \to \mathbb{Q}_{\geq 0}$ is a locally constant function, called *the height* of $\mathcal{M}$. We denote by $\mathcal{M}_i$ the kernel of multiplication by π^i. ▲

Remark B.2.

1) For every $i, j \in \mathbb{N}$, we have an exact sequence $0 \to \mathcal{M}_i \xrightarrow{\text{inclusion}} \mathcal{M}_{j+i} \xrightarrow{\pi^i\cdot} \mathcal{M}_j \to 0$.

2) $\mathcal{M}_j$ is finite locally free (flat) $\mathcal{O}$-module scheme over S and has order equal to q^{jh}. ◊

The following remark gives another definition of π-divisible modules:

Remark B.3. Assume that we have a sequence $\mathcal{M}_1 \xrightarrow{\iota_1} \mathcal{M}_2 \xrightarrow{\iota_2} \mathcal{M}_3 \to \cdots$ where the $\mathcal{M}_i$ are finite locally free $\mathcal{O}$-module schemes over S with the following properties:

- the order of $\mathcal{M}_j$ is equal to q^{jh}, with h a fixed locally constant map $S \to \mathbb{Q}_{\geq 0}$,
- the sequences $0 \to \mathcal{M}_j \xrightarrow{\iota_j} \mathcal{M}_{j+1} \xrightarrow{\pi^j} \mathcal{M}_{j+1}$ are exact.

Then, the limit $\varinjlim(\mathcal{M}_n, \iota_n)$ is a π-divisible module over S of height h and for every $j > 0$, $\mathrm{Ker}(\pi^j \cdot) \cong \mathcal{M}_j$. ◊

Remark B.4. Let A be a Henselian local ring and $\mathcal{M}$ a π-divisible formal $\mathcal{O}$-module over A. The connected-étale sequence of $\mathcal{M}$, $0 \to \mathcal{M}^\circ \to \mathcal{M} \to \mathcal{M}^{\text{ét}} \to 0$, as a formal group scheme over A, is in fact a sequence of π-divisible modules over A. We have $(\mathcal{M}^\circ)_n = (\mathcal{M}_n)^\circ$ and $(\mathcal{M}^{\text{ét}})_n = (\mathcal{M}_n)^{\text{ét}}$. ◊

Proposition B.5. *Let $\mathcal{M}$ be an étale π-divisible module over a base scheme S. Then there exists a finite étale cover $T \to S$ such that $\mathcal{M}_T$ is the constant π-divisible module with $\mathcal{M}_{n,T} \cong \underline{(\mathcal{O}/\pi^n)}^h$, where h is the height of $\mathcal{M}$. If S is connected, we can take a connected finite étale cover T.*

Proof. The proof is similar to the case of p-divisible groups. □

Remark B.6. In the mixed characteristic case, any π-divisible module of height h is canonically a p-divisible group of height efh. If the base scheme is the spectrum of a field, then every p-divisible group with an $\mathcal{O}$-action is canonically a π-divisible module. ◊

Definition B.7. Let $\mathcal{M}$ be a π divisible module over S and denote by $\mathcal{M}_n^*$ the Cartier dual of $\mathcal{M}_n$. The inductive system $\mathcal{M}_1^* \to \mathcal{M}_2^* \to \mathcal{M}_3^* \to \ldots$ induced by the homomorphisms $\mathcal{M}_{i+1} \to \mathcal{M}_i$ (cf. Remark B.2), is called the dual π-divisible module of $\mathcal{M}$. ▲

Remark B.8. By functoriality of Cartier duality, the action of $\mathcal{O}$ on $\mathcal{M}_n$ induces an action on $\mathcal{M}_n^*$. It follows that the dual of a π-divisible module is again a π-divisible module of the same height. ◊

Let $\mathcal{M}$ be a π-divisible module over a field k of characteristic p. For every $i > 0$, we have exact sequences $0 \to \mathcal{M}_i \xrightarrow{\iota} \mathcal{M}_{i+1} \xrightarrow{\pi^i} \mathcal{M}_{i+1}$ and $\mathcal{M}_{i+1} \xrightarrow{\pi^i} \mathcal{M}_{i+1} \xrightarrow{\Pi} \mathcal{M}_i \to 0$. Therefore, if k is perfect, whether we use the covariant or contravariant Dieudonné theory (and if we denote by D the Dieudonné functor), we have an injection $\eta : D(\mathcal{M}_i) \hookrightarrow D(\mathcal{M}_{i+1})$ and a surjection $\zeta : D(\mathcal{M}_{i+1}) \twoheadrightarrow D(\mathcal{M}_i)$ such that the compositions

$$D(\mathcal{M}_i) \overset{\eta}{\hookrightarrow} D(\mathcal{M}_{i+1}) \overset{\zeta}{\twoheadrightarrow} D(\mathcal{M}_i) \quad \text{and} \quad D(\mathcal{M}_{i+1}) \overset{\zeta}{\twoheadrightarrow} D(\mathcal{M}_i) \overset{\eta}{\hookrightarrow} D(\mathcal{M}_{i+1})$$

are multiplication by π and we have the following exact sequences:

$$D(\mathcal{M}_{i+1}) \xrightarrow{\pi^i} D(\mathcal{M}_{i+1}) \xrightarrow{\zeta} D(\mathcal{M}_i) \to 0 \quad \text{and} \quad 0 \to D(\mathcal{M}_i) \xrightarrow{\eta} D(\mathcal{M}_{i+1}) \xrightarrow{\pi^i} D(\mathcal{M}_{i+1}).$$

Definition B.9. Let k be a perfect field of characteristic p and $\mathcal{M}$ a π-divisible module over k.

(i) We define the *Dieudonné module* of a π-divisible module $\mathcal{M}$ to be the inverse limit $\varprojlim_i (D(\mathcal{M}_i), \zeta)$.

It is called the *covariant* (respectively *contravariant) Dieudonné module*, if it is the inverse limit of covariant (respectively contravariant) Dieudonné modules.

(ii) The morphism induced on the Dieudonné module of $\mathcal{M}$ by the Frobenius morphisms (respectively Verschiebungen) of $D(\mathcal{M}_i)$ is called *the Frobenius* (respectively *Verschiebung*) and is denoted by F (respectively V). ▲

Lemma B.10. *Let $\mathcal{O} = \mathbb{F}_q[\![\pi]\!]$ and $\mathcal{M}$ be a π-divisible module over a perfect field k containing $\mathbb{F}_q$. Then the contravariant Dieudonné module of $\mathcal{M}$, $D(\mathcal{M})$, is a finite free module over $\mathcal{O} \otimes_{\mathbb{F}_q} k \cong k[\![\pi]\!]$.*

Proof. Let us write $\mathcal{M} = \bigcup \mathcal{M}_n$ where $\mathcal{M}_n$ are kernels of π^n and write D (respectively D_n) for $D(\mathcal{M})$ (respectively $D(\mathcal{M}_n)$). Then $D = \varprojlim D_n$. The Dieudonné module D_n is finite over $W(k)$, but $p \cdot \mathcal{M} = 0$, which implies that $p \cdot D = 0$ and thus D_n is finite over $W(k)/p = k$. Let $d_1, \cdots, d_r$ be elements in D whose images in D_1 is a basis over k and define a morphism $k[\![\pi]\!]^r \to D$ by sending basis elements to d_i. This morphism induces morphisms $(k[\![\pi]\!]/(\pi^n))^r \to D/\pi^n D \cong D_n$ which are surjective (since modulo π they are surjective) an so, being an inverse limit of surjective morphisms, $k[\![\pi]\!]^r \to D$ is surjective. This implies that D is a finite module over $k[\![\pi]\!]$. The action of π on $\mathcal{M}$ is surjective and therefore its action on D is injective. It follows that D is a torsion-free $k[\![\pi]\!]$-module and hence is free over it, since $k[\![\pi]\!]$ is a principal ideal domain. □

Theorem B.11. *Finite dimensional π-divisible modules are formally smooth.*

Proof. Note that we may assume that the base scheme is an algebraically closed field k.

Let $\mathcal{M}$ be a π-divisible module over k. If $\mathcal{M}$ is étale it is smooth as well. So, we can assume that the characteristic of k is p. Since k is perfect, $\mathcal{M}$ splits into the étale and connected factors and so, we may assume that $\mathcal{M}$ is connected. As for connected formal schemes, being smooth is equivalent to the Frobenius morphism being an epimorphism, we will show that the Frobenius morphism is an epimorphism. In the mixed characteristic case, by Remark B.6, $\mathcal{M}$ is a p-divisible group. Since the multiplication by p factors through Frobenius and multiplication by p is an epimorphism, $F_{\mathcal{M}}$ is an epimorphism too. Now, assume that $\mathcal{O} = \mathbb{F}_q[\![\pi]\!]$. By Lemma B.10, the contravariant Dieudonné module of $\mathcal{M}$, $D := D(\mathcal{M})$ is a finite free $k[\![\pi]\!]$-module. Denote by σ the endomorphism of $k[\![\pi]\!]$ which is identity on π and Frobenius on k. Since the action of π on $\mathcal{M}$ is a morphism of formal group schemes, it commutes with the Frobenius of $\mathcal{M}$ and thus the Frobenius morphism of D is σ-linear. The cokernel of the Frobenius of D has finite dimension over k (equal to the dimension of $\mathcal{M}$). It follows that Frobenius of D is injective[1] and therefore the Frobenius of $\mathcal{M}$ is an epimorphism. Hence the smoothness of $\mathcal{M}$. □

Remark B.12. What we have shown in the last theorem is that the Frobenius morphism of the contravariant Dieudonné module of a π-divisible module is injective. We will see in the next lemma that similarly, the Verschiebung of the covariant Dieudonné module of a π-divisible module is injective as well. ♢

[1] note that D is a $k[[\pi]]$-module and the dimension of $k[[\pi]]$ over k is infinite.

Lemma B.13. *Let $\mathcal{M}$ be a finite dimensional π-divisible module over a perfect field k of characteristic p, then the Verschiebung morphism of the covariant Dieudonné module of $\mathcal{M}$ is injective.*

Proof. Let D and resp. D_i denote the covariant Dieudonné module of $\mathcal{M}$ and resp. of $\mathcal{M}_i$. Denote by K_i the kernel of the Verschiebung of D_i. The tangent space $\mathcal{L}ie(\mathcal{M}_i)$ is canonically isomorphic to the cokernel of the Verschiebung and since D_i is a $W(k)$-module of finite length, its kernel and cokernel have the same length and therefore, we have $\dim_k K_i = \dim_k \mathcal{L}ie(\mathcal{M}_i)$. So $\dim_k K_i \leq \dim_k \mathcal{L}ie(\mathcal{M}) < \infty$. Since the inclusion $\eta : D_i \hookrightarrow D_{i+j}$ is compatible with the Verschiebungen, it induces a morphism between kernels of V, which we denote also by $\eta : K_i \hookrightarrow K_{i+j}$. Similarly, the epimorphism $\zeta : D_{i+j} \twoheadrightarrow D_i$ induces a morphism $\zeta : K_{i+j} \to K_i$, and as we have seen before (Remark B.8), the composition $K_{i+j} \xrightarrow{\zeta} K_i \xhookrightarrow{\eta} K_{i+j}$ is the multiplication by π^j. Now, since the dimension of K_i is bounded above, and we have inclusions $K_i \hookrightarrow K_{i+1}$, there exists n_0, such that for all $i \geq n_0$, we have $\dim_k K_i = \dim_k K_{n_0}$ and so $\eta : K_i \hookrightarrow K_{i+j}$ is an isomorphism for any j and any $i \geq n_0$. We claim that $\zeta : K_{i+n_0} \to K_i$ are zero for all $i \geq 1$. Indeed, the composition $K_{i+n_0} \to K_i \hookrightarrow K_{i+n_0}$ is the multiplication by π^{n_0}, and so, composed with the inclusion $K_{n_0} \hookrightarrow K_{i+n_0}$ is zero (note that $K_{n_0} \subset D_{n_0}$ and D_{n_0} is killed by π^{n_0}), which implies that the composition $K_{n_0} \hookrightarrow K_{i+n_0} \to K_i$ is zero ($K_i \hookrightarrow K_{i+n_0}$ is injective). But, the morphism $K_{n_0} \hookrightarrow K_{i+n_0}$ is actually an isomorphism by the choice of n_0, and therefore the morphism $K_{i+n_0} \to K_i$ is zero and the claim is proved. For every i, we have an exact sequence $0 \to K_i \to D_i \xrightarrow{V} D_i$. Taking the inverse limit over i with the transition morphisms ζ, and recalling that D is the inverse limit of D_i, we obtain the exact sequence $0 \to \varprojlim K_i \to D \xrightarrow{V} D$. But, since for every $i \geq 1$, the transition morphism $K_{i+n_0} \to K_i$ is zero, it follows that the inverse limit $\varprojlim K_i$ is trivial, and hence $V : D \to D$ is injective. □

Theorem B.14. *Let S be a scheme and $\mathcal{M}$ a π-divisible module over S. Then the height $h(\mathcal{M}) : S \to \mathbb{Q}_{\geq 0}$ takes integer values.*

Proof. Since the height is invariant under base change, we may assume that S is the spectrum of an algebraically closed field k. The order of finite group schemes is multiplicative with respect to exact sequences and it follows from Remark B.4 that the height of $\mathcal{M}$ is the sum of the heights of its connected and étale parts. So, we prove the statement for étale and connected π-divisible modules separately. Assume that $\mathcal{M}$ is étale. Then by Remark B.4, $\mathcal{M}_1$ is étale, and so $\mathcal{M}_1$ is a constant group scheme, of order equal to the order of the group $\mathcal{M}_1(k)$, which is a module over $\mathcal{O}/\pi = \mathbb{F}_q$. The height of $\mathcal{M}$ is by definition equal to $\log_q |\mathcal{M}_1| = \log_q |\mathcal{M}_1(k)|$. Being a vector space over $\mathbb{F}_q$, the order of $\mathcal{M}_1(k)$ is a power of q and therefore the height of $\mathcal{M}$ is a natural number. Now, assume that $\mathcal{M}$ is connected and so k has characteristic p. By Theorem B.11, $\mathcal{M}$ is smooth and therefore it is a commutative formal Lie group. Again, we consider the problem in mixed and equal characteristic cases separately. If $\mathcal{O}$ has characteristic zero, by Remark B.6, $\mathcal{M}$ is a p-divisible group of height $efh(\mathcal{M})$. Note that for p-divisible groups, the height is always a natural number and so $efh(\mathcal{M}) \in \mathbb{N}$. From Theorem 1 of [8], we know that the height of $\mathcal{M}$, regarded as a p-divisible group, is divisible by the degree of the extension $K/\mathbb{Q}_p$, which is ef. Therefore, $h(\mathcal{M})$ is a natural number. Now, assume that $\mathcal{O} = \mathbb{F}_q[\![\pi]\!]$. The Dieudonné module of $\mathcal{M}$, $D := D(\mathcal{M})$, is a finitely generated module over the ring $k \otimes_{\mathbb{F}_p} \mathcal{O}$ (see Lemma B.10). This ring is isomorphic to the product $\prod_{j \bmod f} k \otimes_{\mathbb{F}_q} \mathcal{O}$, and the isomorphism is given explicitly by

$$\theta : k \otimes_{\mathbb{F}_p} \mathbb{F}_q[\![\pi]\!] \to \prod_{j \bmod f} k \otimes_{\mathbb{F}_q} \mathbb{F}_q[\![\pi]\!] = \prod_{j \bmod f} k[\![\pi]\!]$$

$$x \otimes (\sum_{i=0}^{\infty} a_i \pi^i) \mapsto (x^{p^j} \otimes (\sum_{i=0}^{\infty} a_i \pi^i))_j = (\sum_{i=0}^{\infty} (a_i x^{p^j}) \pi^i)_j.$$

This decomposition of $k \otimes_{\mathbb{F}_p} \mathcal{O}$ gives a decomposition of D, so $D = \prod C_j$ where each C_j is a module over $k[\![\pi]\!]$. We know from Lemma B.10 that D is a finite free module over $k[\![\pi]\!]$, it follows that C_j are also finite free modules over $k[\![\pi]\!]$. Let us denote by h_j the rank of C_j over $k[\![\pi]\!]$. Then the rank of D is $\sum_{j \bmod f} h_j$. Using the isomorphism θ one can show that the Frobenius morphism of D induces $F : C_j \to C_{j-1}$. As we have seen in the proof of Theorem B.11, it follows that the Frobenius of D is injective, therefore, we have $h_j \leq h_{j-1}$. This being true for every j mod f, we conclude that $h_j = h_{j'}$ for all j and j' in $\mathbb{Z}/f\mathbb{Z}$. Call this common number h. The rank of D as a module over $k[\![\pi]\!]$ is then equal to fh. But we know from Dieudonné theory that the rank of D (which is equal to the length of $D(\mathcal{M}_1)$ over k), is equal to $\log_p |\mathcal{M}_1| = fh(\mathcal{M})$. Hence, $h(\mathcal{M}) = h$ and it is a natural number. □

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