The Notion of Resolution in Model Categories

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We begin with a number of definitions that we will use throughout the article. For the following definitions we fix a model category $\mathcal{C}$ and we denote the initial object and terminal object by 0 and $*$ respectively.

Definition 1. *Let $f : A \to B$ be a morphism in $\mathcal{C}$, the* kernel *of f, denoted by $ker(f)$, is defined to be the pullback of the morphisms $0 \to B \xleftarrow{f} A$, i.e. we have a commutative diagram in which $ker(f)$ is a pullback:*

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\big\uparrow{\scriptstyle i} & & \big\uparrow \\
ker(f) & \longrightarrow & 0
\end{array}
$$

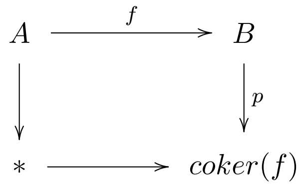
Definition 2. *Let E be an arbitrary but fixed functorial factorization of every morphism $g : X \to Y$ into $X \xrightarrow{i_g} E(g) \xrightarrow{p_g} Y$, where i_g is a trivial cofibration and p_g is a fibration, that we fix throughout the article. The* homotopy kernel *of the morphism $f : A \to B$, denoted by $hker(f)$, is defined to be the pullback of the diagram $A \xrightarrow{f} B \xleftarrow{p_b} E(B)$ where $0 \xrightarrow{b} B$ is the unique morphism from the initial object to B. We may show the situation by the following commutative diagram:*

$$
\begin{array}{ccccc}
A & \xrightarrow{f} & B & \xleftarrow{b} & \\
\big\uparrow{\scriptstyle i} & & \big\uparrow{\scriptstyle p_b} & & \\
hker(f) & \xrightarrow[j]{} & E(B) & \xleftarrow[\sim]{i_b} & 0
\end{array}
$$

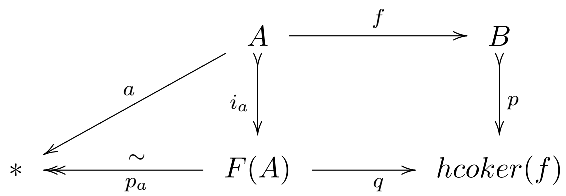
Remark 3. Since the fibrations are preserved under pullbacks, the morphism $hker(f) \xrightarrow{i} A$ is a fibration, hence the notation $hker(f) \twoheadrightarrow A$.

We can dualize these definition as follows:

Definition 4. *Let $f : A \to B$ be a morphism in $\mathcal{C}$, the* cokernel *of f, denoted by $coker(f)$, is defined to be the pushout of the morphisms $* \leftarrow A \xrightarrow{f} B$, i.e. we have a commutative diagram in which $coker(f)$ is a pushout:*

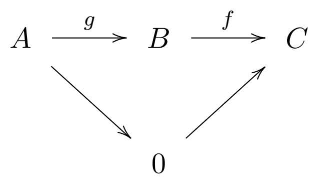


Definition 5. *Let F be an arbitrary but fixed functorial factorization of every morphism $g : X \to Y$ into $X \xrightarrow{i_g} F(g) \xrightarrow{p_g} Y$, where i_g is a cofibration and p_g is a trivial fibration, that we fix throughout the article. The* homotopy cokernel *of the morphism $f : A \to B$, denoted by $hcoker(f)$, is defined to be the pushout of the diagram $F(A) \overset{i_a}{\longleftarrow} A \xrightarrow{f} B$ where $A \xrightarrow{a} *$ is the unique morphism from A to the terminal object. We may show the situation by the following commutative diagram:*

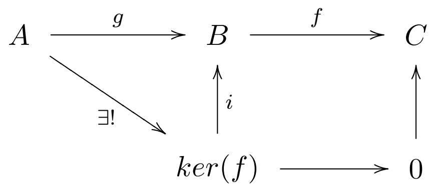


Remark 6. Since the cofibrations are preserved under pushouts, the morphism $B \xrightarrow{p} hcoker(g)$ is a cofibration, hence the notation $B \rightarrowtail hcoker(g)$.

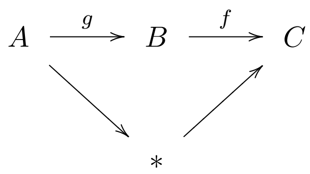
Suppose that we have a sequence of morphisms $A \xrightarrow{g} B \xrightarrow{f} C$ in $\mathcal{C}$ such that their composition factorizes through the initial object, i.e. there exists a morphism $A \to 0$ that makes the following diagram commute:



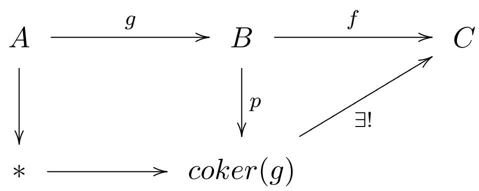
then by the universal property of pullbacks there exists a unique morphism $A \to ker(f)$ such that the following diagram is commutative:



Similarly suppose that we have the same sequence of morphisms in $\mathcal{C}$ such that their composition factorizes, this time, through the terminal object, i.e. there exists a morphism $* \to C$ that makes the following diagram commutative:



then by the universal property of pushouts there exists a unique morphism $coker(g) \to C$ such that the following diagram is commutative:



Remark 7. In the case that the category $\mathcal{C}$ possesses a *zero object*, i.e. an object that is in the same time an initial and a terminal object, the existence of a factorization of the composition of this sequence through the zero object amounts to saying that the composition of the unique morphisms from A to this object and from this object to C is equal to the composition of g and f.

Definition 8. *A sequence of morphisms*

$$C_* : \quad \cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots$$

is called a kernel complex *if every composition $d_n \circ d_{n+1}$ factorizes through the initial object. A kernel-complex composed of three objects, is called a* short kernel-complex. *Similarly we call the sequence*

$$C_* : \quad \cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots$$

cokernel-complex *if every composition* $d_n \circ d_{n+1}$ *factorizes through the terminal object. A cokernel-complex composed of three objects is called* short cokernel-complex. *If the category* $\mathcal{C}$ *has a zero object then kernel-complexes (resp. short kernel-complexes) and cokernel-complexes (resp. short cokernel-complexes) coincide and we call such a sequence a* complex *(resp.* short complex.*)*

Definition 9. *We say that the sequence of morphisms* $A \xrightarrow{g} B \xrightarrow{f} C$ *in* $\mathcal{C}$ *is* kernel-exact *at* B *if it is a short kernel-complex and the unique morphism from* A *to* $ker(f)$ *is a trivial fibration, i.e. we have a commutative diagram:*

$$\begin{array}{ccccc} A & \xrightarrow{g} & B & \xrightarrow{f} & C \\ & \searrow^{\sim} & \uparrow{\scriptstyle i} & & \uparrow \\ & & ker(f) & \longrightarrow & 0 \end{array}$$

A sequence

$$C_* : \quad \cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots$$

is kernel-exact if it is a kernel-complex and is exact at C_n *for every* n*, i.e. for every* n *the short kernel-complex*

$$C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1}$$

is kernel-exact at C_n*.*

Definition 10. *We say that the sequence of morphisms* $A \xrightarrow{g} B \xrightarrow{f} C$ *in* $\mathcal{C}$ *is* cokernel-exact *at* B *if it is a short cokernel-complex and the unique morphism from* $coker(g)$ *to* C *is a trivial cofibration, i.e. we have a commutative diagram:*

$$\begin{array}{ccccc} A & \xrightarrow{g} & B & \xrightarrow{f} & C \\ \downarrow & & \downarrow{\scriptstyle p} & \nearrow_{\sim} & \\ * & \longrightarrow & coker(g) & & \end{array}$$

In the same way we say that a sequence

$$C_* : \quad \cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots$$

is cokernel-exact if it is a cokernel-complex and is exact at C_n for every n, i.e. for every n the short cokernel-complex

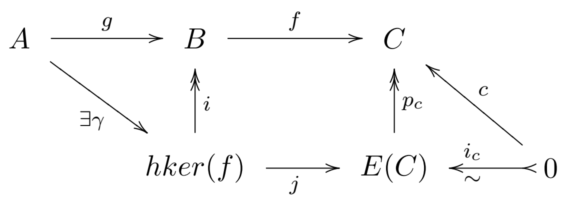
$$C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1}$$

is cokernel-exact at C_n.

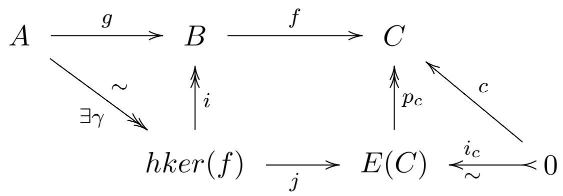
Remark 11. In the two previous definitions, if the category has a zero object, the unique induced morphisms do not depend on the choice of the factorization of the composition $f \circ g$ through the zero object, since there exist one such factorization.

Definition 12. *We say that the sequence of morphisms $A \xrightarrow{g} B \xrightarrow{f} C$ in $\mathcal{C}$ is a* short weak kernel-complex *if there is a morphism $A \xrightarrow{\gamma} hker(f)$ with $i \circ \gamma = g$. We call it* homotopy-kernel-exact *at B if there is a trivial fibration $A \xrightarrow{\gamma} hker(f)$ with $i \circ \gamma = g$. We may show these situations as follows:*

Short weak kernel-complex:



Homotopy kernel-exact:



In the same fashion we say that a sequence

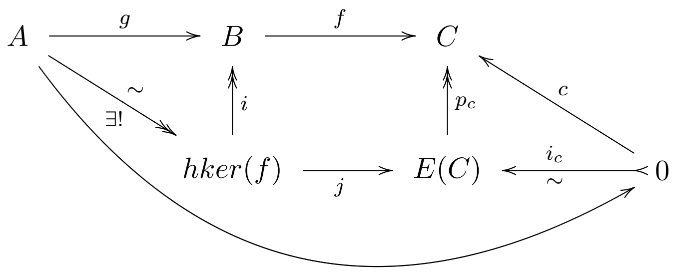
$$C_* : \quad \cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots$$

is a weak kernel-complex *(resp.* homotopy-kernel-exact*) if for every n, the sequence*

$$C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1}$$

is a short weak kernel-complex (resp. homotopy-kernel-exact).

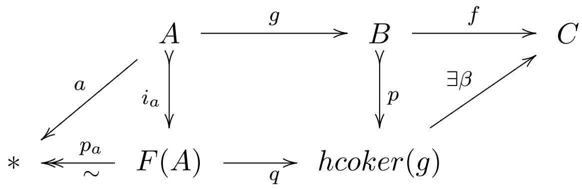
Remark 13. If the composition of f and g factorizes through the initial object, then we may compose the morphism from A to 0 with the morphism i_c to form a morphism from A to $E(C)$, and from the universal property of pullbacks there is automatically a morphism from A to $hker(f)$, i.e. we have a commutative diagram:



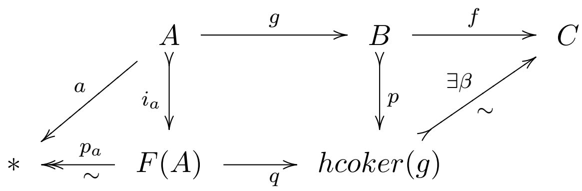
so any kernel-complex is a weak kernel-complex.

Definition 14. *We say that the sequence of morphisms* $A \xrightarrow{g} B \xrightarrow{f} C$ *in* $\mathcal{C}$ *is a* short weak cokernel-complex *if there is morphism* $hcoker(g) \xrightarrow{\beta} C$ *such that* $\beta \circ p = f$*. We call it* homotopy-cokernel-exact *at* B *if there is a trivial cofibration* $hcoker(g) \xrightarrow{\beta} C$ *such that* $\beta \circ p = f$*. We may show the situation as follows:*

Short weak cokernel-complex:



Homotopy cokernel-exact:



In the same way we say that a sequence

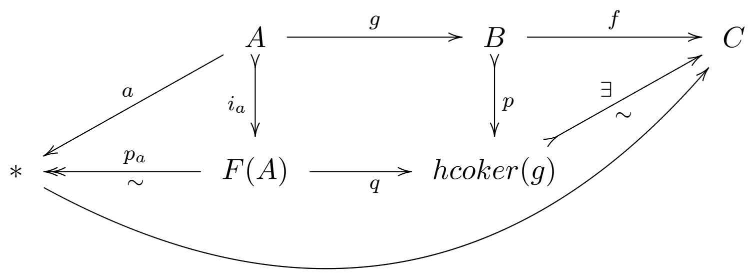
$$C_* : \quad \cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots$$

is a weak cokernel-complex *(resp.* homotopy-cokernel-exact*) if for every* n*, the sequence*

$$C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1}$$

is a short weak cokernel-complex (resp. homotopy-cokernel-exact).

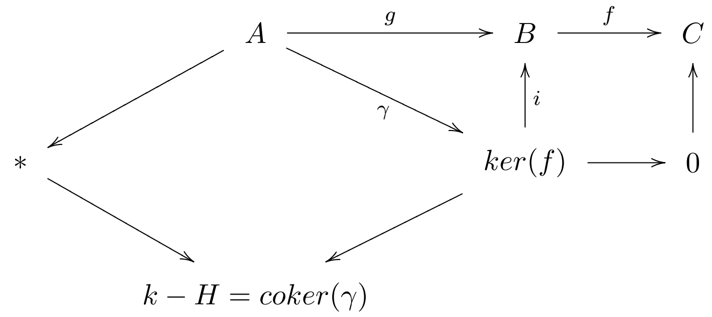
Remark 15. A similar remark as remark 13 goes as well for the case of a factorization through the terminal object, so if the composition of f and g factorizes through the terminal object, then we may precompose the morphism from $*$ to C with the morphism p_a to form a morphism from $F(A)$ to C, and from the universal property of pushouts there is automatically a morphism from $hcoker(g)$ to C, i.e. we have a commutative diagram:



and so any cokernel-complex is a weak-cokernel-complex.

Remark 16. Note that in Definition 12 (resp. 14) the morphism g (resp. f) is a fibration (resp. a cofibration), since they are compositions of two morphisms of this types.

Definition 17. *Suppose that we have a short kernel-complex* $A \xrightarrow{g} B \xrightarrow{f} C$ *in* $\mathcal{C}$*, then the* kernel-homology *of this sequence (at* B*), denoted by* $k-H(A \xrightarrow{g} B \xrightarrow{f} C)$*, is defined to be the cokernel of the induced morphism from* A *to* $ker(f)$*, i.e. we have the following commutative diagram:*



And for any kernel-complex

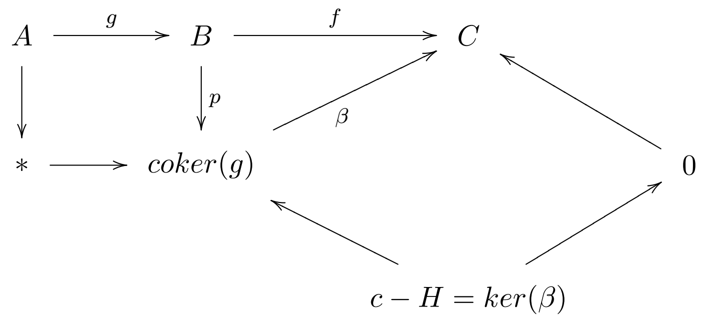
$$C_* : \quad \cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots$$

the n-th kernel-homology*, denoted $k - H_n(C_*)$, is defined to be the kernel-homology of the short kernel-complex*

$$C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1}.$$

We define the dual as follows:

Definition 18. *Suppose that we have a short cokernel-complex $A \xrightarrow{g} B \xrightarrow{f} C$ in the category $\mathcal{C}$, then the* cokernel-homology *of this sequence (at B), denoted by $c - H(A \xrightarrow{g} B \xrightarrow{f} C)$, is defined to be the kernel of the induced morphism from $coker(g)$ to C, i.e. we have the following commutative diagram:*



And for any cokernel-complex

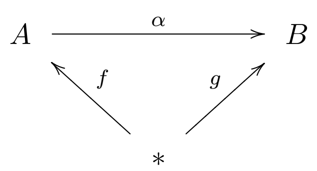
$$C_* : \quad \cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots$$

the n-th cokernel-homology*, denoted* c$-H_n(C_*)$*, is defined to be the cokernel-homology of the short cokernel-complex*

$$C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1}.$$

As we have remarked in Remark 9, if the category $\mathcal{C}$ has a zero object, which we denote again by 0, then the kernel-homology and cokernel-homology are independent from the choice of factorizations through the zero object, again, since there exist a unique factorization through the zero object. We will often suppose that our model categories has a zero object and the following arguments show that this assumption is not very restrictive.

Definition 19. *Given a model category $\mathcal{C}$, define $\mathcal{C}_*$ to be the category under the terminal object $*$, i.e. an object of $\mathcal{C}_*$ is a morphism $* \xrightarrow{f} A$ of $\mathcal{C}$, often written (A, f), and a morphism from (A, f) to (B, g) is a morphism $A \xrightarrow{\alpha}$ such that the following diagram is commutative:*



We think of (A, f) as an object A with a basepoint f. There is an obvious functor $V : \mathcal{C} \longrightarrow \mathcal{C}_*$ that takes A to $A_+ = A \amalg *$, with the basepoint $* \to A \amalg *$, that we write as well $*$. There is forgetful functor $U : \mathcal{C}_* \longrightarrow \mathcal{C}$ which takes (A, f) to A.

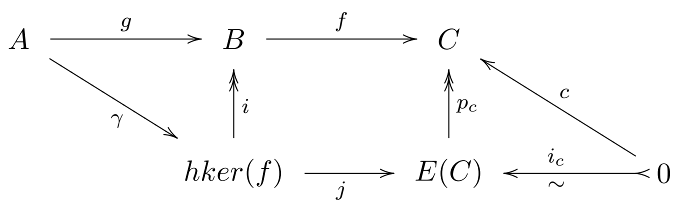
Proposition 20. *The functor V defined above, which adds a disjoint basepoint is left adjoint to the forgetful adjoint U and defines a faithful (but not full) embedding of $\mathcal{C}$ into the pointed category $\mathcal{C}_*$. If $\mathcal{C}$ is already pointed, these functors define an equivalence of categories between $\mathcal{C}$ and $\mathcal{C}_*$.*

PROOF. See [Hovey] p.4 . □

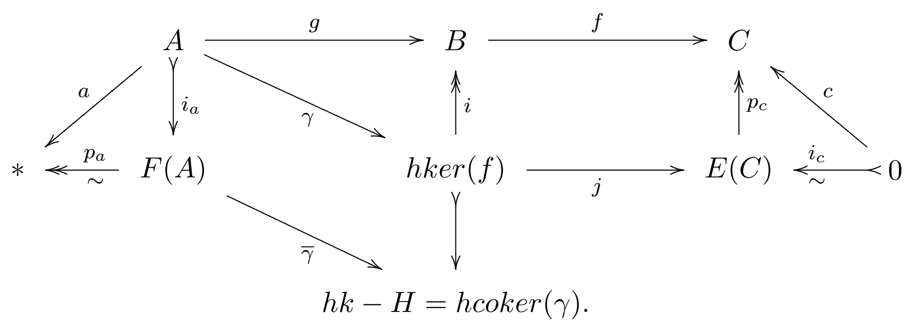
Proposition 21. *Suppose that $\mathcal{C}$ is a model category. Define a morphism in $\mathcal{C}_*$ to be a cofibration (fibration, weak equivalence) if and only if $U(f)$ is a cofibration (fibration, weak equivalence) in $\mathcal{C}$. Then $\mathcal{C}_*$ is a model category. Furthermore any Quillen adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ induces a Quillen adjunction $F_* : \mathcal{C}_* \rightleftarrows \mathcal{D}_* : G_*$, and $F_*(A_+)$ is naturally isomorphic to $F(A)_+$. This correspondence is functorial.*

PROOF. See [Hovey] pp.5-15 . □

Definition 22. *Given a short weak kernel-complex* $A \xrightarrow{g} B \xrightarrow{f} C$*, so we have a commutative diagram described in Definition 12:*



Now the homotopy kernel-homology *of this sequence, denoted* $hk{-}H(A \xrightarrow{g} B \xrightarrow{f} C)$ *is defined to be the homotopy cokernel of* γ*, i.e,* $hk{-}H = hcoker(\gamma)$*. Schematically we have:*



Now for any weak kernel-complex

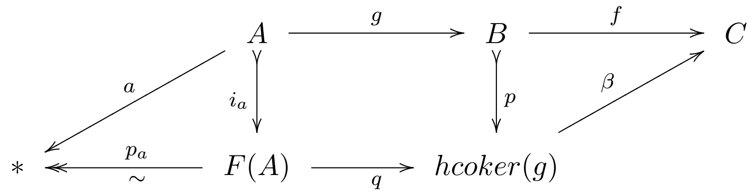
$$C_* : \quad \cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots$$

the $n-th$ homotopy kernel-homology*, denoted* $hk - H_n(C_*)$*, is defined to be the homotopy kernel-homology of the short weak kernel-complex*

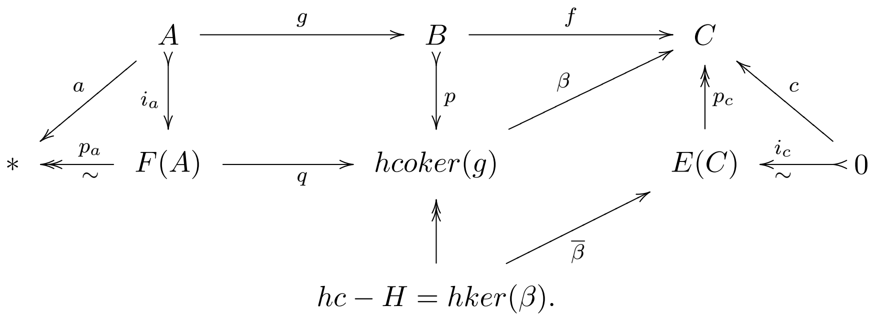
$$C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1}.$$

We have the dual definition as follows

Definition 23. *Given a short weak cokernel-complex* $A \xrightarrow{g} B \xrightarrow{f} C$*, so we have a commutative diagram described in Definition 14:*



Now the homotopy cokernel-homology *of this sequence, denoted* $hc - H(A \xrightarrow{g} B \xrightarrow{f} C)$ *is defined to be the homotopy kernel of* β*, i.e,* $hc - H = hker(\beta)$*. Schematically we have:*



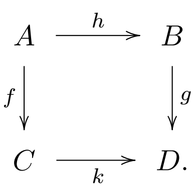
Now for any weak cokernel-complex

$$C_* : \quad \cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots$$

the $n - th$ homotopy cokernel-homology*, denoted* $hc - H_n(C_*)$*, is defined to be the homotopy cokernel-homology of the short weak cokernel-complex*

$$C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1}.$$

Definition 24. *Let* $\mathcal{C}$ *be a model category and consider the following commutative square:*



If the square is a pushout, then the morphism g *is called the* pushout of f along h*. If the square is a pullback, then the morphism* f *is called the* pullback of g along k.

Definition 25. *Let $\mathcal{C}$ be a model category.*

(i) *The model category $\mathcal{C}$ is called* left proper *if every pushout of a weak equivalence along a cofibration is a weak equivalence.*

(ii) *The model category $\mathcal{C}$ is called* right proper *if every pullback of a weak equivalence along a fibration is a weak equivalence.*

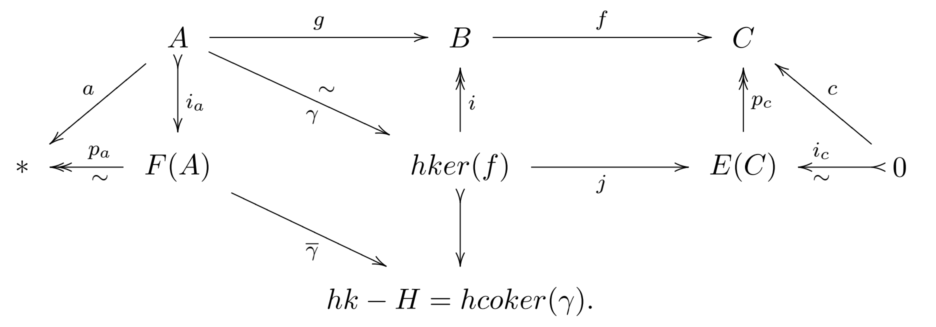
(iii) *The model category $\mathcal{C}$ is called* proper *if it is both left proper and right proper.*

Proposition 26. *Suppose that $\mathcal{C}$ is a left proper model category and that*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is a homotopy-kernel-exact. Then the homotopy kernel-homology of this sequence is weak equivalent to the terminal object.

PROOF. Since the sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is homotopy kernel-exact, we have the following commutative diagram:



Now, using the fact that $\mathcal{C}$ is left proper and that $\overline{\gamma}$ is the pushout of a weak equivalence (γ) along a cofibration (i_a), we conclude that $\overline{\gamma}$ is also a weak equivalence. The composition of $\overline{\gamma}$ with the unique morphism $hk - H \xrightarrow{p_{hk-H}} *$ is the unique morphism from $F(A)$ to the terminal object, namely p_a, which is a weak equivalence. Using the 2-of-3 property implies that p_{hk-H} is a weak equivalence. □

A similar argument proves the following proposition:

Proposition 27. *Suppose that $\mathcal{C}$ is a right proper model category and that*

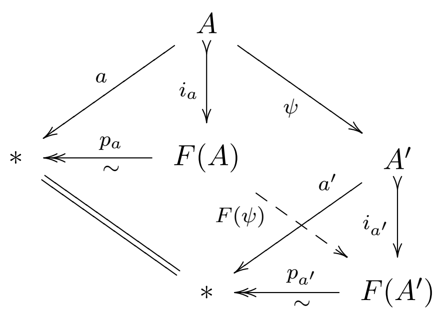
$$A \xrightarrow{f} B \xrightarrow{g} C$$

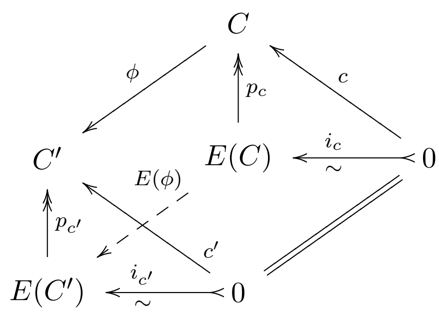
is a homotopy-cokernel-exact. Then the homotopy cokernel-homology of this sequence is weak equivalent to the initial object.

PROOF. □

To be able to compare homologies of different complexes we need the notion of morphisms between complexes. But before, we need some propositions.

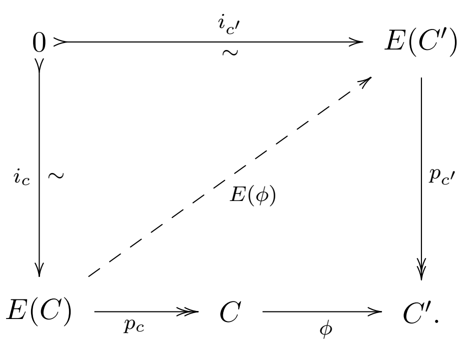
Proposition 28. *Let $A \xrightarrow{\psi} A', C \xrightarrow{\phi} C'$ be morphisms of the category $\mathcal{C}$. Then there are morphisms $F(A) \xrightarrow{F(\psi)} F(A')$, $E(C) \xrightarrow{E(\phi)} E(C')$ that make the following diagrams commutative:*





Remark 29. *Note that the choice of the morphism $E(\phi) : E(C) \to E(C')$ is not unique, and we fix it for any morphism.*

PROOF. We prove the existence of $E(\phi)$, the other one is similar. Since we have the following commutative diagram in which i_c is a trivial cofibration and $p_{c'}$ a fibration, from the axioms of model categories there is a lifting in the diagram that makes the whole diagram commute:

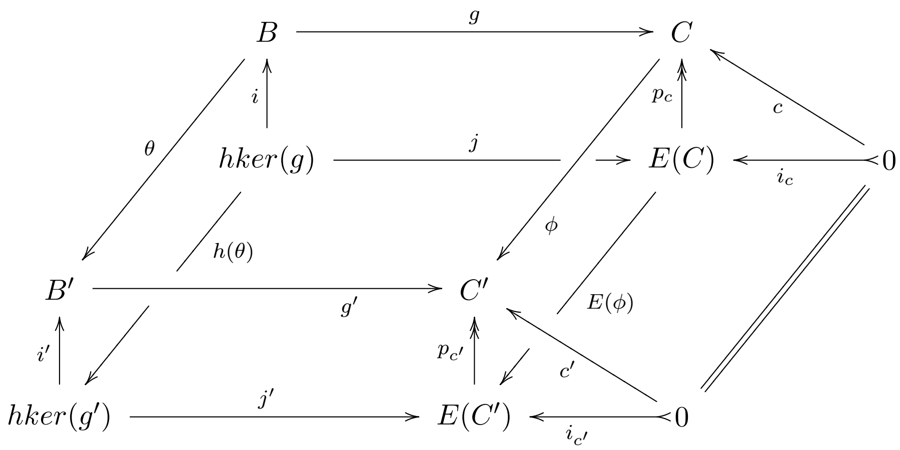


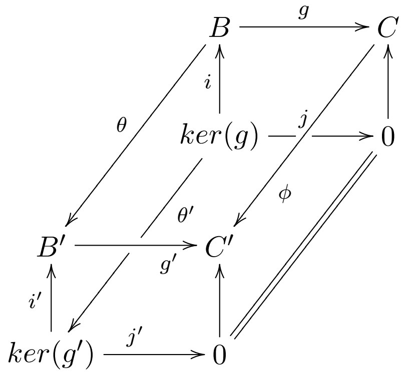
□

Proposition 30. *Suppose that $g : B \to C$, $g' : B' \to C'$, $\theta : B \to B'$ and $\phi : C \to C'$ be morphisms in $\mathcal{C}$ such that $\phi \circ g = g' \circ \theta$, i.e. the following diagram commutes:*

$$\begin{array}{ccc} B & \xrightarrow{g} & C \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\phi} \\ B' & \xrightarrow[g']{} & C'. \end{array}$$

Then there are unique morphisms $h(\theta) : hker(g) \to hker(g')$ and $\theta' : ker(g) \to ker(g')$ making the following diagrams commute:





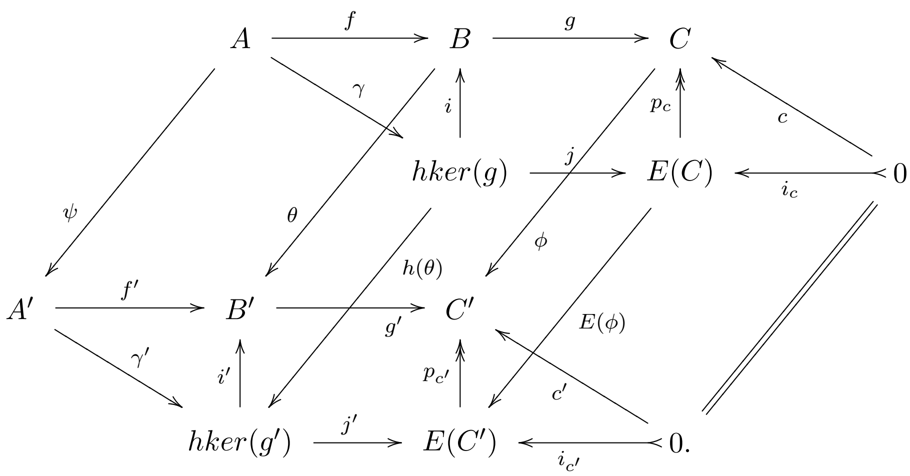
where the morphism $E(\phi)$ is given from the previous proposition.

Proof. For the first diagram, we have two morphisms $\theta \circ i : hker(g) \to B'$ and $E(\phi) \circ j : hker(g) \to E(C')$ such that $g' \circ \theta \circ i = p_{c'} \circ E(\phi) \circ j$. So by the universal property of pullbacks there is a unique morphism $hker(g) \to hker(g')$ that makes the diagram commute. A similar argument applies to the other diagram. $\square$

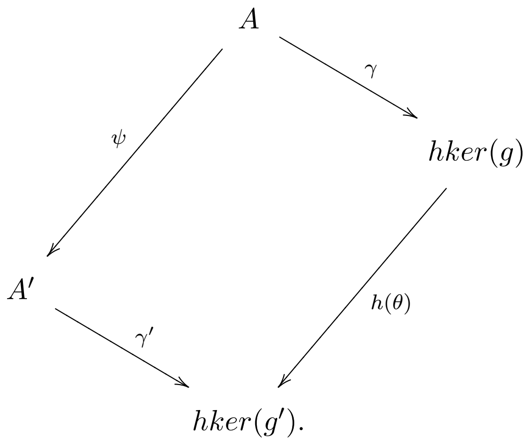
Now suppose that we have two short weak kernel-complexes $A \xrightarrow{f} B \xrightarrow{g} C$ and $A' \xrightarrow{f'} B' \xrightarrow{g'} C'$ with morphisms $A \xrightarrow{\psi} A'$, $B \xrightarrow{\theta} B'$ and $C \xrightarrow{\phi} C'$ such that the following diagram commutes:

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \phi} \\ A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C'. \end{array}$$

Then from the previous proposition and the definition of such complexes we have a (not in general commutative!) diagram (which we denote by (★)):



In fact, the only square that doesn't necessarily commute is the following:



Definition 31. *Suppose that we have two short weak kernel-complexes* $A \xrightarrow{f} B \xrightarrow{g} C$ *and* $A' \xrightarrow{f'} B' \xrightarrow{g'} C'$. *Then* a morphism (of such complexes) *between them consists of three morphisms* $A \xrightarrow{\psi} A'$, $B \xrightarrow{\theta} B'$ *and* $C \xrightarrow{\phi} C'$ *such that the last diagram (or equivalently the one coming before, i.e.* (★)*) is commutative.*

A morphism between two weak kernel-complexes

$$C_* : \quad \cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots$$

and

$$C'_* : \quad \cdots \to C'_{n+1} \xrightarrow{d'_{n+1}} C'_n \xrightarrow{d'_n} C'_{n-1} \xrightarrow{d'_{n-1}} \cdots$$

is a collection of morphisms $\{\phi_n : C_n \to C'_n\}$ *such that for every* n, $\phi_{n+1} : C_{n+1} \to C'_{n+1}$, $\phi_n : C_n \to C'_n$ *and* $\phi_{n-1} : C_{n-1} \to C'_{n-1}$ *is a morphism between the two short weak kernel-complexes*

$$C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1}$$

and

$$C'_{n+1} \xrightarrow{d'_{n+1}} C'_n \xrightarrow{d'_n} C'_{n-1}.$$

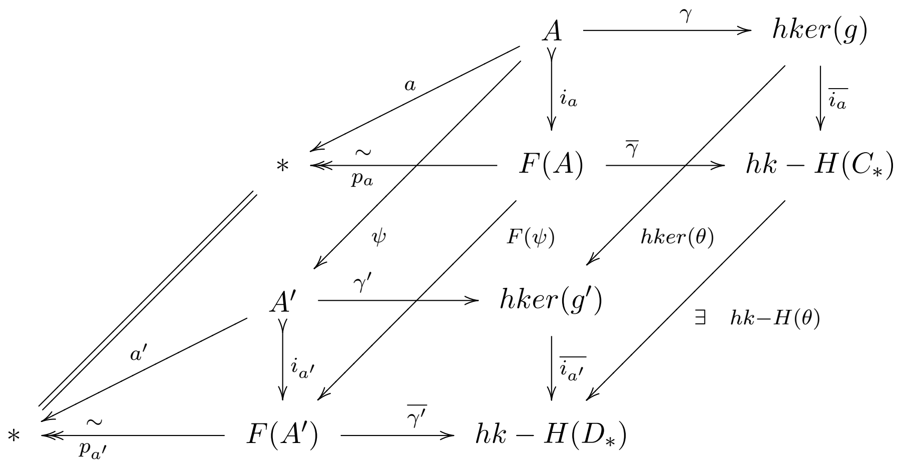
The definition of morphisms between short kernel-complexes is defined similarly; one has only to delete the factorizations of the morphisms from initial object to C *and* C', *and take the kernels instead of homotopy kernels.*

Remark 32. Note that the definition of a morphism between two sequences $A \xrightarrow{f} B \xrightarrow{g} C$ and $A' \xrightarrow{f'} B' \xrightarrow{g'} C'$ depends on the choice of the morphism $E(\phi) : E(C) \to E(C')$.

Proposition 33. *Suppose that we have two short weak kernel-complexes (denoted by C_* and D_*) with a morphism between them:*

$$\begin{array}{ccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
\downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \phi} \\
A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C'.
\end{array}$$

Then there is a morphism between the homotopy kernel-homology of C_* and that of D_* such that with the notations of diagram (★), the following diagram commutes:



where the morphism $F(\psi)$ is given by Proposition 27.

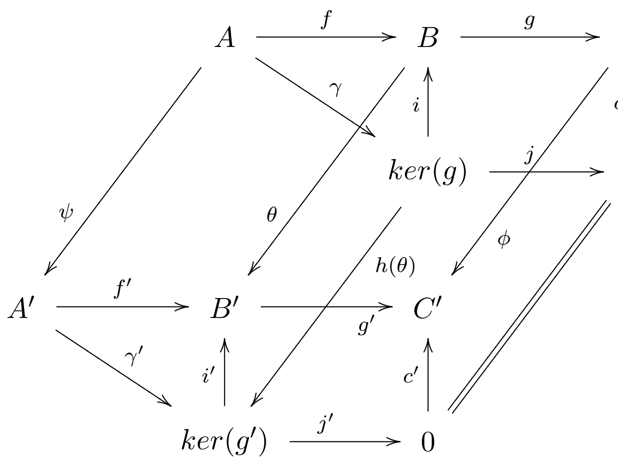
Proof. We have two morphisms $\overline{i_{a'}} \circ hker(\theta) : hker(g) \to hk - H(D_*)$ and $\overline{\gamma'} \circ F(\psi) : F(A) \to hk - H(D_*)$, and the fact that the top (horizontal) square commutes, i.e. $hker(\theta) \circ \gamma = \gamma' \circ \psi$ implies that the precomposition of these two morphisms respectively with γ and i_a coincide and therefore from the universal property of pushouts there is a unique morphism from $hk - H(C_*)$ to $hk - H(D_*)$ with the desired property. □

One can easily dualize these arguments to the case of weak cokernel-complexes to get a similar definition and result for morphisms between such complexes.

Remark 34. Note that if the category has a zero object and if we have a commutative diagram with rows short kernel-complexes:

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \downarrow \psi & & \downarrow \theta & & \downarrow \phi \\ A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C'. \end{array}$$

Then these morphisms are in fact morphisms of complexes, for in the following diagram :



the morphisms $h(\theta) \circ \gamma$ and $\gamma' \circ \psi$ from A to $ker(g')$ are equal after composition with i' and j', for

$$i' \circ h(\theta)\gamma = \theta \circ i \circ \gamma = \theta \circ f = f' \circ \psi = i' \circ \gamma'\psi$$

and the other equality is due to the fact that there is a unique morphism from A to 0. It follows from the universal property of pullbacks that these morphisms are equal, which shows that we have a morphism of complexes.

Definition 35. *Suppose we have an object X in the model category $\mathcal{C}$. A* cofibrant resolution (resp. weak cofibrant resolution) *of X is a kernel-exact (resp. homotopy-kernel-exact) complex*

$$\cdots \longrightarrow Q_{n+1} \xrightarrow{d_{n+1}} Q_n \xrightarrow{d_n} Q_{n-1} \longrightarrow \cdots \longrightarrow Q_0 \xrightarrow{\pi} X.$$

Where Q_i's are cofibrant objects and the morphism $\pi : Q_0 \to X$ is a trivial fibration.

Definition 36. *Suppose we have an object Y in the model category $\mathcal{C}$. A* fibrant resolution (resp. weak fibrant resolution) *of Y is a cokernel-exact (resp. homotopy-cokernel-exact) complex*

$$Y \xrightarrow{\iota} R^0 \cdots \longrightarrow R^{n-1} \xrightarrow{d^{n-1}} R^n \xrightarrow{d^n} R^{n+1} \longrightarrow \cdots .$$

Where R^i's are fibrant objects and the morphism $\iota : Y \to R^0$ is a trivial cofibration.

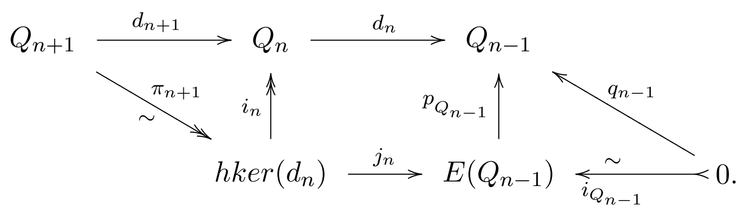
From now on we will consider only the cofibrant or weak cofibrant resolutions and leave out the dual results and considerations, since one has only to add a "co" or delete a "co" and reverse the direction of morphisms to get them !

Proposition 37. *Given an object X of the model category $\mathcal{C}$, then there exists a cofibrant and a weak cofibrant resolution for X.*

Proof. We show firstly the existence of a weak cofibrant resolution. We construct it inductively. For $n = 0$ put $Q_0 \overset{\pi}{\underset{\sim}{\twoheadrightarrow}} X$ a cofibrant replacement of X. Suppose that we have constructed

$$Q_n \xrightarrow{d_n} Q_{n-1} \longrightarrow \cdots \longrightarrow Q_0 \xrightarrow{\pi} X$$

with $n \geq 0$, $Q_{-1} := X$ and $d_0 := \pi$. Define $Q_{n+1} \overset{\pi_{n+1}}{\underset{\sim}{\twoheadrightarrow}} hker(d_n)$ to be a cofibrant replacement of $hker(d_n)$ and $d_{n+1} : Q_{n+1} \to Q_n$ to be the composition of π_{n+1} with the morphism $hker(d_n) \to Q_n$ in the pullback diagram. We have thus the following commutative diagram:



It is now clear from the construction that the resulting complex is in fact a weak cofibrant resolution of X. If we take cofibrant replacement of kernels instead of homotopy kernels, then we will have a cofibrant resolution of X. A piece of this resolution is shown in the following diagram:

$$
\begin{array}{ccccc}
Q_{n+1} & \xrightarrow{d_{n+1}} & Q_n & \xrightarrow{d_n} & Q_{n-1} \\
 & \searrow^{\pi_{n+1}}_{\sim} & \uparrow{\scriptstyle i_n} & & \uparrow{\scriptstyle q_{n-1}} \\
 & & ker(d_n) & \xrightarrow{j_n} & 0.
\end{array}
$$

□

Proposition 38. *Suppose that we have a morphism $\phi : X \to Y$ in the model category $\mathcal{C}$, a weak cofibrant resolution of X:*

$$
Q_* : \quad \cdots \longrightarrow Q_{n+1} \xrightarrow{d_{n+1}} Q_n \xrightarrow{d_n} Q_{n-1} \longrightarrow \cdots \longrightarrow Q_0 \xrightarrow{\pi} X
$$

and a homotopy kernel-exact complex as follows:

$$
P_* : \quad \cdots \longrightarrow P_{n+1} \xrightarrow{d'_{n+1}} P_n \xrightarrow{d'_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \xrightarrow{\pi'} Y.
$$

with $\pi' : P_0 \twoheadrightarrow Y$ a trivial fibration. Then there exists a morphism of weak kernel-complexes $\phi_* : Q_* \to P_*$.

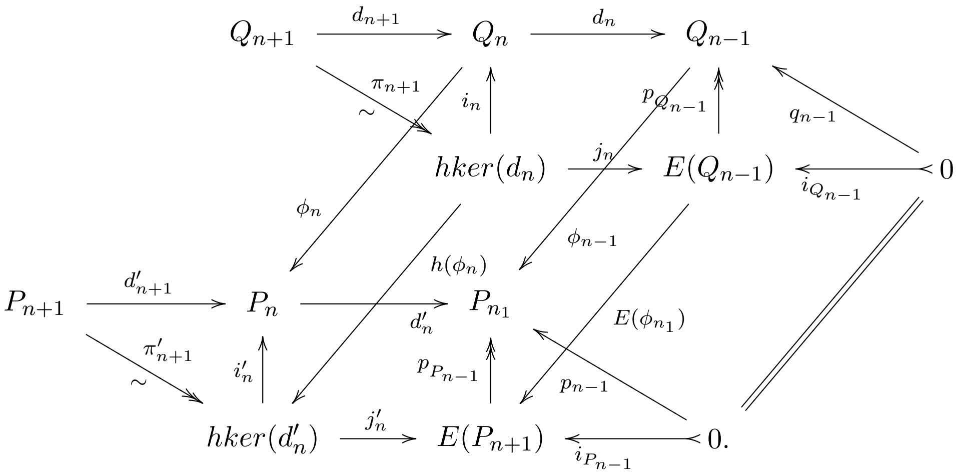
Proof. We will again construct morphisms $\phi_n : Q_n \to P_n$ inductively. We make the following conventions $Q_{-1} = X$, $P_{-1} = Y$, $d_0 : Q_0 \to Q_{-1} = \pi : Q_0 \to X$ and $\phi_{-1} = \phi$. Since the morphisms from the initial object to any object are unique we have a commutative diagram:

$$
\begin{array}{ccccc}
0 & & \xrightarrow{\quad p_0 \quad} & & P_0 \\
\downarrow{\scriptstyle q_0} & & \nearrow^{\exists \phi_0} & & \downarrow{\scriptstyle \pi'}\,\sim \\
Q_0 & \underset{\sim}{\overset{\pi}{\twoheadrightarrow}} & X & \xrightarrow{\phi} & Y.
\end{array}
$$

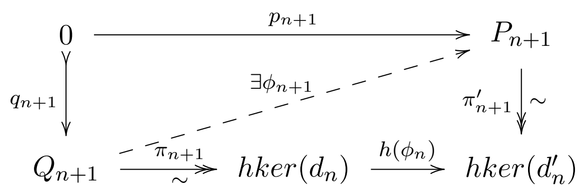
Now from the axioms of model categories there exists a filler in this diagram, i.e. there is a morphism $\phi_0 : Q_0 \to P_0$ that makes the last diagram commute, so we have the following commutative diagram:

$$
\begin{array}{ccc}
Q_0 & \underset{\sim}{\overset{\pi}{\twoheadrightarrow}} & Q_{-1} \\
\downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle \phi_{-1}} \\
P_0 & \underset{\sim}{\overset{\pi'}{\twoheadrightarrow}} & P_{-1}.
\end{array}
$$

Now suppose that we have constructed for any $n \geq 0$ the morphisms $\phi_n : Q_n \to P_n$ that are morphisms of weak kernel-complexes (up to degree n) and so from Proposition 28 we have the following commutative diagram:



and we construct it for $n+1$. As before, we have a commutative diagram:



and we have therefore a filler $\phi_{n+1} : Q_{n+1} \to P_{n+1}$ in the diagram. It implies that we have $\pi'_{n+1} \circ \phi_{n+1} = h(\phi_n) \circ \pi_{n+1}$ and so

$$d'_{n+1} \circ \phi_{n+1} = i'_n \circ \pi'_{n+1} \circ \phi_{n+1} = i'_n \circ d'_n \circ \pi_{n+1} = \phi_n \circ i_n \pi_{n+1} = \phi_n \circ d_{n+1}.$$

It follows that we have morphisms of weak kernel-complexes up to degree $n+1$. $\square$

Again if we replace homotopy kernels by kernels we have a similar result that we mention without proof.

Proposition 39. *Suppose that we have a morphism $\phi : X \to Y$ in the model category $\mathcal{C}$, a cofibrant resolution of X:*

$$Q_* : \quad \cdots \longrightarrow Q_{n+1} \xrightarrow{d_{n+1}} Q_n \xrightarrow{d_n} Q_{n-1} \longrightarrow \cdots \longrightarrow Q_0 \xrightarrow{\pi} X$$

and a kernel-exact complex as follows:

$$P_* : \quad \cdots \longrightarrow P_{n+1} \xrightarrow{d'_{n+1}} P_n \xrightarrow{d'_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \xrightarrow{\pi'} Y.$$

with $\pi' : P_0 \twoheadrightarrow Y$ a trivial fibration. Then there exists a morphism of kernel-complexes $\phi_* : Q_* \to P_*$.

Proposition 40. *Let $\mathcal{C}$ be a right proper model category. If the vertical morphisms in the commutative diagram:*

$$\begin{array}{ccccc} X & \longrightarrow & Z & \longleftarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Z' & \longleftarrow & Y' \end{array}$$

are weak equivalences and at least one morphism in each row is a fibration, then the morphism of pullbacks $X \times_Z Y \to X' \times_{Z'} Y'$ induced by the diagram is a weak equivalence.

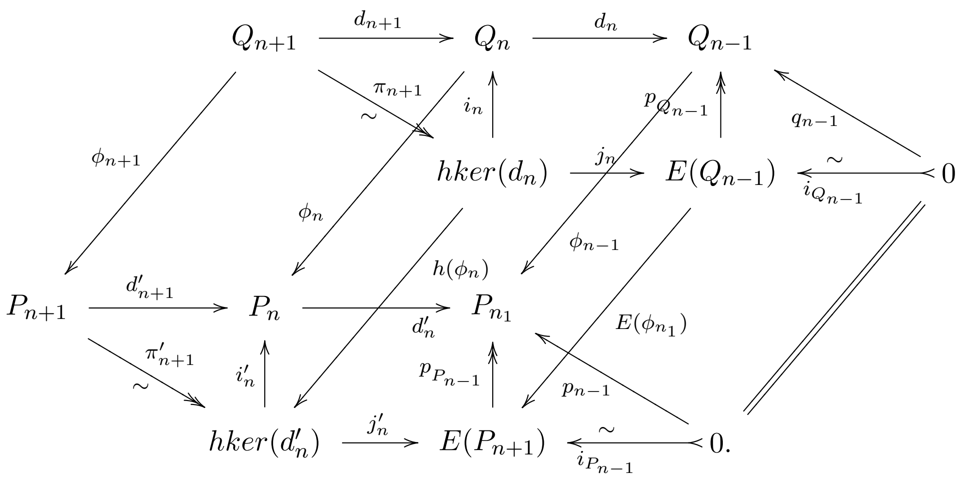
Proof. See [Hirschhorn] p.247 . □

Proposition 41. *Suppose that we have a weak equivalence $X \xrightarrow[\sim]{f} Y$ in a right proper model category $\mathcal{C}$ and weak cofibrant resolutions Q_* and P_* of X and Y respectively with a morphism, ϕ_*, of weak kernel complexes between them, i.e we have the following commutative diagram:*

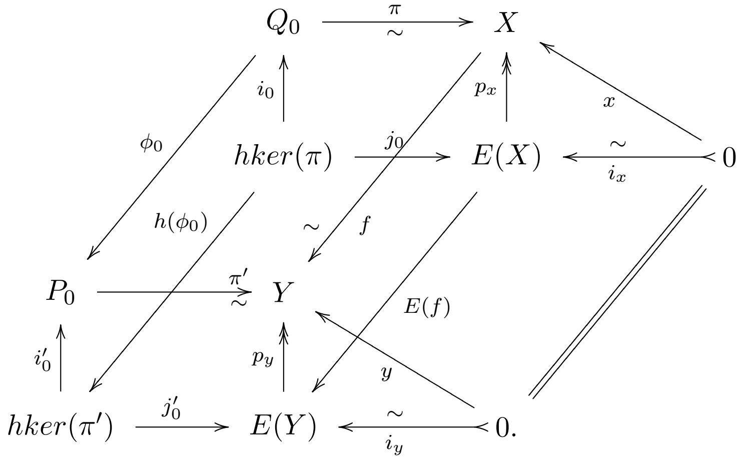
$$\begin{array}{ccccccccccc} \cdots \longrightarrow & Q_{n+1} & \xrightarrow{d_{n+1}} & Q_n & \xrightarrow{d_n} & Q_{n-1} & \xrightarrow{d_{n-1}} \cdots \longrightarrow & Q_0 & \xrightarrow[\sim]{\pi} & X \\ & \downarrow{\scriptstyle \phi_{n+1}} & & \downarrow{\scriptstyle \phi_n} & & \downarrow{\scriptstyle \phi_{n-1}} & & \downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle f}\,\wr \\ \cdots \longrightarrow & P_{n+1} & \xrightarrow{d'_{n+1}} & P_n & \xrightarrow{d'_n} & P_{n-1} & \xrightarrow{d'_{n-1}} \cdots \longrightarrow & P_0 & \xrightarrow[\sim]{\pi'} & Y \end{array}$$

where the morphisms ϕ_i form a morphism of weak kernel complexes. Then for any $n \geq 0$, the morphism ϕ_n is a weak equivalence.

Proof. Since ϕ_* is a morphism of weak kernel complexes, with our convention that $Q_{-1} = X$, $P_{-1} = Y$ and $\phi_{-1} = f$ we have for any $n \geq 0$ the following commutative diagram (denoted (★)):



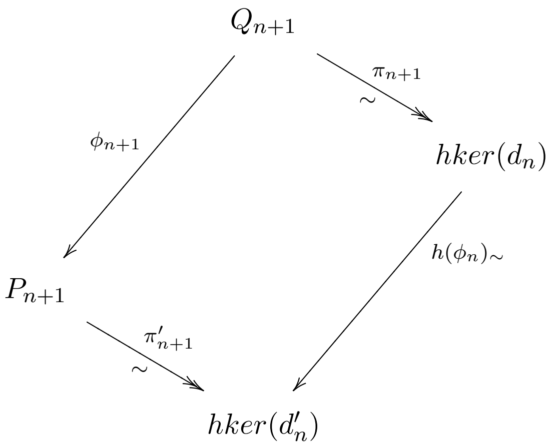
We show by induction on $n \geq 0$ that ϕ_n and $h(\phi_n)$ are weak equivalences. For $n = 0$ we have the following commutative diagram:



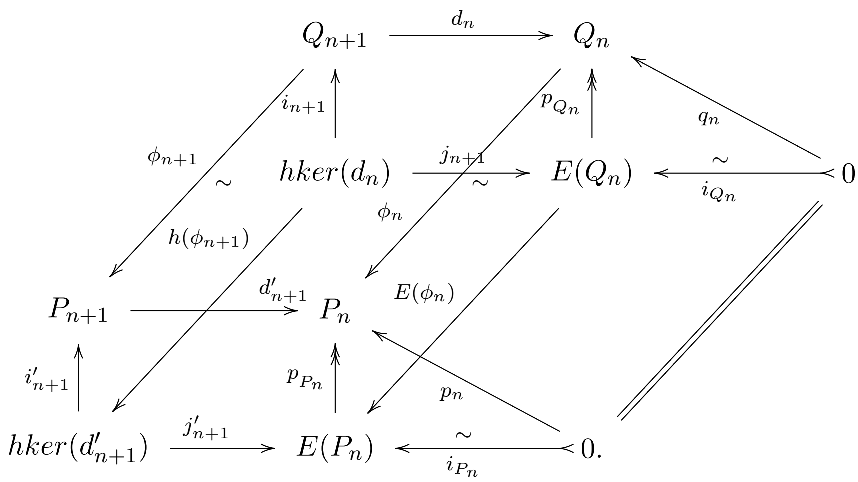
We have $E(f) \circ i_x = i_y$ and since i_x and i_y are weak equivalences the 2-of-3 property of model categories implies that $E(f)$ is also a weak equivalence. Since we have also $f \circ \pi = \pi' \circ \phi_0$ the same argument shows that ϕ_0 is a weak equivalence. Now the fact that $\mathcal{C}$ is right proper implies that j_0 and j'_0 are weak equivalences and one more application of 2-of-3 property of model categories shows that $h(\phi_0)$ is a weak equivalence.

Now suppose we know that for $n \geq -1$, ϕ_n and $h(\phi_n)$ are weak equivalences, then we show that ϕ_{n+1} and $h(\phi_{n+1})$ are weak equivalences too.

From the commutativity of the square



in the diagram (★) and again 2-of-3 property, it follows that ϕ_{n+1} is a weak equivalence. To show that $h(\phi_{n+1})$ is a weak equivalence, consider the following commutative diagram:



since $E(\phi_n) \circ i_{Q_n} = i_{P_n}$ and i_{Q_n} and i_{P_n} are weak equivalences, $E(\phi_n)$ has to be a weak equivalence as well, so we can extract the following diagram from the last diagram:

$$\begin{array}{ccccc}
Q_{n+1} & \xrightarrow{d_{n+1}} & Q_n & \xleftarrow{p_{Q_n}} & E(Q_n) \\
\sim\downarrow \phi_{n+1} & & \sim\downarrow \phi_n & & \sim\downarrow E(\phi_n) \\
P_{n+1} & \xrightarrow{d'_{n+1}} & P_n & \xleftarrow{p_{P_n}} & E(P_n).
\end{array}$$

Note that $hker(d_n)$ and $hker(d'_n)$ are by definition the pullbacks of this diagram and that the morphism $h(\phi_{n+1}) : hker(d_n) \to hker(d'_n)$ is exactly the morphism induced by this diagram, we can therefore apply the last proposition and deduce that $h(\phi_{n+1})$ is a weak equivalence. This achieves the proof of the proposition. □

Remark 42. This proposition shows in particular that any two weak cofibrant resolutions of an object are degreewise weak equivalent. This will allow us to make definitions which are invariant under weak equivalences.

Remark 43. Suppose that we have a (covariant) functor $F : \mathcal{C} \to \mathcal{D}$ from a right proper model category $\mathcal{C}$ to an abelian category $\mathcal{D}$ such that the image of any weak equivalence under F is an isomorphism in $\mathcal{D}$ and that initial objects in $\mathcal{C}$ map to zero objects in $\mathcal{D}$. Then the last proposition let us define the derived functors of F in the following sense: take an object X in $\mathcal{C}$ and a weak cofibrant resolution

$$Q_* : \quad \cdots \longrightarrow Q_{n+1} \xrightarrow{d_{n+1}} Q_n \xrightarrow{d_n} Q_{n-1} \longrightarrow \cdots \longrightarrow Q_0 \xrightarrow{\pi} X$$

of X, now apply the functor F on this resolution to get the following sequence

$$F(Q_*) : \quad \cdots \longrightarrow F(Q_{n+1}) \xrightarrow{F(d_{n+1})} F(Q_n) \xrightarrow{F(d_n)} F(Q_{n-1}) \longrightarrow \cdots$$

$$\cdots \longrightarrow F(Q_0) \xrightarrow{F(\pi)} F(X).$$

We claim that this sequence is in fact a complex, i.e. $F(d_n) \circ F(d_{n+1}) = 0$ for $n \geq 0$, where $d_0 = \pi$. Indeed, since Q_* is a homotopy-kernel complex, for any $n \geq 1$ we have a commutative diagram

$$\begin{array}{ccccccc}
Q_{n+1} & \xrightarrow{d_{n+1}} & Q_n & \xrightarrow{d_n} & Q_{n-1} & & \\
& \searrow^{\pi_{n+1}}_{\sim} & \uparrow{\scriptstyle i_n} & & \uparrow{\scriptstyle p_{Q_{n-1}}} & \nwarrow^{q_{n-1}} & \\
& & hker(d_n) & \xrightarrow{j_n} & E(Q_{n-1}) & \underset{i_{Q_{n-1}}}{\overset{\sim}{\longleftarrow}} & 0.
\end{array}$$

Applying the functor F on it we obtain the following commutative diagram

$$\begin{array}{ccccccc}
F(Q_{n+1}) & \xrightarrow{F(d_{n+1})} & F(Q_n) & \xrightarrow{F(d_n)} & F(Q_{n-1}) & & \\
& \searrow^{F(\pi_{n+1})}_{\cong} & \uparrow{\scriptstyle F(i_n)} & & \uparrow{\scriptstyle F(p_{Q_{n-1}})} & \nwarrow^{F(q_{n-1})} & \\
& & F(hker(d_n)) & \xrightarrow{F(j_n)} & F(E(Q_{n-1})) & \underset{F(i_{Q_{n-1}})}{\overset{\cong}{\longleftarrow}} & 0.
\end{array}$$

Since the object $F(E(Q_{n-1}))$ is isomorphic to a zero object it is also a zero object and we see that the composition $F(d_n) \circ F(d_{n+1})$ factorizes through a zero object and hence is zero.

Now define the *n-th left derived functor* of F denoted by $\mathcal{L}_n F$ to be the n-th homology of the resulting complex after replacing the first term of it, namely $F(X)$, by 0. The previous proposition states that the left derived functors are well defined up to isomorphism. If the functor F is contravariant we denote its n-th left derived functor by $\mathcal{L}^n F$. Carrying out the same construction in the case where $\mathcal{C}$ is a left proper model category and using weak fibrant resolutions gives rise to *right derived functor* of F and we denote it by $\mathcal{R}_n F$ ($\mathcal{R}^n F$ resp.) if F is covariant (contravariant resp.).

Example 44. As an example of the previous remark we can look at the functors $\pi_n : \mathcal{T}op_* \to \mathfrak{Ab}$ $(n \geq 2)$, from the category of based topological spaces to the category of abelian groups that send any space X to its n-th homotopy group. Since for any weak equivalence $f : X \to Y$ between two based topological spaces X and Y we have that $\pi_n(f) : \pi_n(X) \to \pi_n(Y)$ is an isomorphism and that the homotopy groups of the singleton $\{*\}$ are all trivial, we can apply the last remark to define the derived functors $\mathcal{L}_m(\pi_n)$ ($\mathcal{R}_m(\pi_n)$ resp.) of π_n that we call *n-th cofibrant homology (n-th fibrant homology resp.)* functors and we denote $\mathcal{L}_m(\pi_n)(X)$ ($\mathcal{R}_m(\pi_n)(X)$ resp.) by $H_{(m,n)}(X)$ ($H^{(m,n)}(X)$ resp.).

Proposition 45. *Suppose that $\mathcal{C}$ is a monoidal model category. Then for any cofibrant object C, the functor $C \otimes - : \mathcal{C} \to \mathcal{C}$ is a Quillen functor with right adjoint $Hom_l(C, -)$. Similarly if D is a cofibrant object, the functor $- \otimes D : \mathcal{C} \to \mathcal{C}$ is a Quillen functor with right adjoint $Hom_r(D, -)$. Also, if E is a fibrant object, the functor $Hom_r(-, E) : \mathcal{C} \to \mathcal{C}^{op}$ is a Quillen functor with right adjoint $Hom_l(-, E)$, where $\mathcal{C}^{op}$ is given the opposite model structure.*

PROOF. See [Hovey] p.108 . □

Definition 46. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between model categories. We say that F is* exact *if F preserves limits and colimits and that $F(Cof_{\mathcal{C}}) \subseteq Cof_{\mathcal{D}}, F(Fib_{\mathcal{C}}) \subseteq Fib_{\mathcal{D}}$ and $F(WE_{\mathcal{C}}) \subseteq WE_{\mathcal{D}}$.*

Example 47. *Suppose that $\mathcal{D}$ is a very small category, i.e. it has finitely many objects and morphisms. Suppose also that $\mathcal{C}$ is a model category. Then the category $\mathcal{C}^{\mathcal{D}}$ inherits two different model structures. These two structures have the property that for any given $d \in D$ and any morphism $\tau : F \to G \in Mor(\mathcal{C}^{\mathcal{D}})$, we have:*

$$\tau \text{ is a cofibration} \Rightarrow \tau(d) : F(d) \to G(d) \text{ is a cofibration}$$

$$\tau \text{ is a fibration} \Rightarrow \tau(d) : F(d) \to G(d) \text{ is a fibration}$$

$$\tau \text{ is a weak equivalence} \Rightarrow \tau(d) : F(d) \to G(d) \text{ is a weak equivalence.}$$

Thus, we have for any $d \in \mathcal{D}$ an exact functor $ev_d : \mathcal{C}^{\mathcal{D}} \to \mathcal{C}$ that sends any $F : \mathcal{D} \to \mathcal{C}$ to $F(d)$.

Proposition 48. *If $F : \mathcal{C} \to \mathcal{D}$ is a exact functor between model categories, then the image of any kernel-exact complex (homotopy kernel-exact, cokernel-exact, homotopy cokernel-exact complex resp.) in $\mathcal{C}$ is a kernel-exact complex (homotopy kernel-exact, cokernel-exact, homotopy cokernel-exact complex resp.) in $\mathcal{D}$.*

Proof. The proof is a direct application of the exactness of the functor F. □

Definition 49. *Let X, Y be two objects in a monoidal model category $\mathcal{C}$. Take a weak cofibrant resolution*

$$Q_* : \quad \cdots \longrightarrow Q_{n+1} \xrightarrow{d_{n+1}} Q_n \xrightarrow{d_n} Q_{n-1} \longrightarrow \cdots \longrightarrow Q_0 \xrightarrow{\pi} X$$

of X and apply the functor $- \otimes Y$ on it to get the complex

$$Q_* \otimes Y : \quad \cdots \longrightarrow Q_{n+1} \otimes Y \xrightarrow{d_{n+1}\otimes Y} Q_n \otimes Y \xrightarrow{d_n \otimes Y} Q_{n-1} \otimes Y \longrightarrow \cdots$$

$$\cdots \longrightarrow Q_0 \otimes Y \xrightarrow{\pi \otimes Y} X \otimes Y.$$

The $Tor^n_{\mathcal{C}}(X, Y)$ is defined to be the n-th homotopy kernel-homology of this complex.

Remark 50. Note that the definition of $Tor^n_{\mathcal{C}}(X, Y)$ depends *a priori* on the choice of the resolution Q_*, since the resulting morphisms between two homologies are not in general weak equivalences.

Definition 51. *Let $\mathcal{C}$ be a model category with a zero object* 0 *and* X *an object in* $\mathcal{C}$*. The* cofibrant (fibrant resp.) dimension *of* X *denoted by* $cdim_{\mathcal{C}}(X)(fdim_{\mathcal{C}}(X)$ *resp.) is the minimum of the lengths of cofibrant (fibrant resp.) resolutions of* X*, where the length of the sequence*

$$Q_* : \quad \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow Q_n \xrightarrow{d_n} Q_{n-1} \longrightarrow \cdots \longrightarrow Q_0 \xrightarrow{\pi} X$$

is n*. If there is no sequence with finite length then this dimension is by convention* ∞*.*

Remark 52. Note that if X is a cofibrant object we have not in general that $cdim(X) = 0$.

Proposition 53. *If* $F : \mathcal{C} \to \mathcal{D}$ *is an exact functor, then for any* $X \in \mathcal{C}$ *we have that* $cdim_{\mathcal{D}}(F(X)) \leq cdim_{\mathcal{C}}(X)$*. The same inequality holds for fibrant dimensions too.*

PROOF. It follows at once from the previous proposition that any resolution of X in $\mathcal{C}$ gives rise to a resolution of $F(X)$ in $\mathcal{D}$ by applying F on it. The result follows immediately. □

Example 54. *As we have seen in last example, if* $\mathcal{D}$ *is a very small category then for any* $d \in \mathcal{D}$ *we have an exact functor* $ev_d : \mathcal{C}^{\mathcal{D}} \to \mathcal{C}$*. Using the last proposition we have for any* $d \in \mathcal{D}$ *and any* $F \in \mathcal{C}^{\mathcal{D}}$*, that* $cdim_{\mathcal{C}}(F(d)) = cdim_{\mathcal{C}}(ev_d(F)) \leq cdim_{\mathcal{C}^{\mathcal{D}}}(F)$*. It implies that* $cdim_{\mathcal{C}^{\mathcal{D}}}(F) \geq max\{cdim_{\mathcal{C}}(F(d)) | d \in \mathcal{D}\}$*.*

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