Hilbert Modular Forms

A Continuation

Adib Abdollahi

May 2023

Plan for the talk

- Recall the definition of automorphic forms and representations

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- Definition of Hilbert modular forms

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- Definition of Hilbert modular forms
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- Address some questions from before

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- *ϕ is of moderate growth.*

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$$\int_{[N]} \phi(ng)dn = 0$$

where $[N] = N(F)\backslash N(\mathbb{A})^1$ for all $N = R_u(P)$ unilpotent radical of parabolic subgroup P and $g \in G(\mathbb{A})$. The subspace of caspidal forms is denoted by $\mathcal{A}_0(G)$.

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Definition

An irreducible admissible $G(\mathbb{A}_f) \times (\mathfrak{g}, K_\infty)$-module is called an automorphic representation (resp. cuspidal automorphic representation) of G if it is a subquotient of $\mathcal{A}(G)$ (resp. $\mathcal{A}_0(G)$).

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- ϕ *in* U*-invariant space* $\mathcal{A}(G, U) := \mathcal{A}(G)^U$.
- *For each* $g_f \in G(\mathbb{A}_f)$, *the function* $\phi(g_f, -) : G(F_\infty) \to \mathbb{C}$ *when restricted to positive determinant elements has the form*

$$\gamma = (\gamma_1, \ldots, \gamma_d) \in G^+(F_\infty) \mapsto f(g_f, \gamma) \prod_{n=1}^{d} det(\gamma_n)^{-k_n/2} j(\gamma_n, i)^{k_n}$$

and factors through a holomorphic function on $G^+(F_\infty)/F_\infty^+ K_\infty^+ \simeq \mathscr{H}_F$.

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We can modify the definition to allow certain type of characters.Let $M_k(U)$ (resp. $S_k(U)$) denote the space of holomorphic (resp. cuspidal) Hilbert modular forms. Bu admissibility of automorphic reps we get that these spaces are finite dimensional (c.f. Harish-Chandra theorem for ideals of finite codim. in $\mathfrak{Z}(\mathfrak{g})$).

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It has a basis $[KgK]$ formed by characteristic functions of double cosets $KgK \in K \backslash G(\mathbb{A}_f)/K$. Let $KgK = \coprod g_i K$. Hecke operator $[KgK]$ sends $f \in M_k(K)$ to $\sum_i f(g_i -)$. This is independent from the representatives g_i's. This gives an action of $\mathcal{H}(G, K)$ on $M_k(K)$ and cuspidal forms are stable under this action.

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Let ϖ_v be a uniformizer for F_v. Hecke operator $[K\left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_v \end{smallmatrix}\right)K]$ is denoted as T_v if K_v is hyperspecial and U_v otherwise. Let $\mathcal{H}^{\mathrm{nr}}(G, K)$ be the unramified Hecke algebra defined as

$$\bigotimes_v{}' \mathcal{H}(G(F_v), K_v)$$

where the restricted tensor product is taken over finite places v such that K_v is hyperspecial. It is commutative.

Old and New forms

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Theorem (Strong multiplicity one theorem for GL_2)

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By letting S be the set of ramifications of $\Pi(f)$ and archimedean places, we conclude that the $\Pi(f)$ is determined by

$$\Pi^S(f) = \bigoplus_{v \quad \text{unramified}}' \Pi_v(f)$$

The representation $\Pi_v(f)$ is uniquely determined by the action of local Hecke algebra $\mathcal{H}(G(F_v), K_v)$ on $\Pi_v(f)^{K_v}$. Since $\Pi_v(f)$ is unramified, we know that $\Pi_v(f)^{K_v}$ is one dimensional and each Hecke operator acts by an scalar, its eigenvalue.

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The subspace $S_k^{\mathrm{new}}(K_1(\mathcal{N}))[\lambda(\Pi)]$ *is non-zero*

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Theorem

There is one to one correspondence between

- *Isomorphism classes of cuspidal automorphic representations Π of GL_2/F with conductor $\mathcal{N}$ and Π_∞ holomorphic discrete series of weight k.*
- *Cuspidal Hilbert modular eigenform newforms over F of level $K_1(\mathcal{N})$ and weight k.*

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Conjecture (Langlands reciprocity (very crude form))

Let π be a (nice) automorphic representation of $G(\mathbb{A}_F)$. Then there exist a (nice) L-parameter $\rho_\pi : \mathcal{L}_F \to {}^LG$ such that L-function of π matches with the L-function of ρ_π and vise versa.

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Conjecture (Langlands functoriality (very crude form))

Let G , H be (connected) reductive groups over F with a L-map ${}^LH \to {}^LG$. Then there is a recipe for transferring a (nice) automorphic representations of $H(\mathbb{A}_F)$ to (nice) automorphic representations of $G(\mathbb{A}_F)$ such that we have a explicit relationship for their L-functions.

Questions from previous discussions

Proposition

Langlands reciprocity $\Rightarrow$ *Langlands functoriality*

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Proof.

Let π_H be an automorphic rep. of H. By reciprocity it corresponds to a L-parameter $\mathcal{L}_F \to {}^L H$. The composition $\mathcal{L}_F \to {}^L H \to {}^L G$ is again a L-parameter for G. Again by reciprocity, there exists an automorphic rep. π_G of G which corresponds to this L-parameter. This is the functorial transform of π_H to π_G. □

Here we use the many details and properties of reciprocity conjecture.

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