AGNT Essentials - problems set 2

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Problem 1 *(For those who have seen the concept of localization before) Verify that* $A \to S^{-1}A$ *satisfies the following universal property:* $S^{-1}A$ *is initial among* A*-algebras,* B*, where every element of* S *is sent to an invertible element in* B*.*

Problem 2 *If* X *is a topological space, show that fiber products always exist in the category of open sets of* X*, by describing what a fibered product is.*

Problem 3 *If* Z *is the final object in category* $\mathcal{C}$*, and* $X, Y \in \mathcal{C}$*. Then, assuming all relevant (fiber)products exist, show that* $X \times_Z Y = X \times Y$

Problem 4 *If the two squares in the following commutative diagram are Cartesian, show that the "outside rectangle" (involving U, V, Y and Z) is also Cartesian.*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

Problem 5 *Given morphisms* $X_1 \to Y, X_2 \to Y$*, and* $Y \to Z$*,*

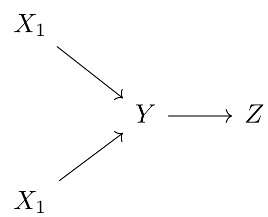
(a) Show that there is a natural morphism $X_1 \times_Y X_2 \to X_1 \times_Z X_2$*, assuming that both fibered products exist.*

(b) Show that if $Y \to Z$ *is a monomorphism, then the morphism you described in the last part is an isomorphism.*

Problem 6 *(a) (The Diagonal-Base-Change Diagram.) Suppose we are given morphisms* $X_1, X_2 \to Y$ *and* $Y \to Z$*. Show that the following is a Cartesian square.*

$$\begin{array}{ccc} X_1 \times_Y X_2 & \longrightarrow & X_1 \times_Z X_2 \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y \times_Z Y \end{array}$$

(b) Solve the last part again by identifying both $X_1 \times_Y X_2$ *and* $Y \times_{Y \times_Z Y} (X_1 \times_Z X_2)$ *as the limit of the diagram.*



Problem 7 *Show that in the category Sets,*

$$\left\{ (a_i)_{i \in \mathcal{I}} \in \prod_i A_i \;:\; F(m)(a_j) = a_k \textit{ for all } m \in Mor_{\mathcal{I}}(j,k) \in Mor(\mathcal{I}) \right\},$$

along with the obvious projection maps to each A_i*, is the limit* $\lim\limits_{\mathcal{I}} A_i$*.*

In the next problem, you will prove the following proposition:

Proposition 1 *Let* $\mathcal{A}$ *be a category.*

(a) If $\mathcal{A}$ *has all products and equalizers then* $\mathcal{A}$ *has all limits.*

(b) If $\mathcal{A}$ *has binary products, a terminal object and equalizers then* $\mathcal{A}$ *has finite limits.*

Problem 8

(a) Let $\mathcal{A}$ *be a category with all products and equalizers. Let* $D : \boldsymbol{I} \to \mathcal{A}$ *be a diagram* $\mathcal{A}$*. Define maps*

$$\prod_{I \in \boldsymbol{I}} D(I) \overset{s}{\underset{t}{\rightrightarrows}} \prod_{J \xrightarrow{u} K \textit{ in } \boldsymbol{I}} D(K)$$

as follows: given $J \xrightarrow{u} K$ *in* $\boldsymbol{I}$*, the u-component of s is the composite*

$$\prod_{I \in \boldsymbol{I}} D(I) \xrightarrow{pr_J} D(J) \xrightarrow{Du} D(K)$$

(where pr denotes a product projection), and the u-component of t is pr_K*. Let* $L \xrightarrow{p} \prod_{I \in \boldsymbol{I}} D(I)$ *be the equalizer of s and t, and write* p_I *for the* $I-$*component of* p*. Show that* $(L \xrightarrow{p_I} D(I))_{I \in \boldsymbol{I}}$ *is a limit cone on D, thus proving the part (a) of the proposition.*

(b) Adapt the argument to prove the part (b) of the proposition.

Problem 9 *Prove that a category with pullbacks and a terminal object has all finite limits.*

Problem 10

(a) Prove that in the category of monoids, the inclusion $(\mathbb{N}, +, 0) \hookrightarrow (\mathbb{Z}, +, 0)$ *is epic, even though it is not surjective.*

(b) Prove that in the category of rings, the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ *is epic, even though it is not surjective.*