AGNT Essentials - problems set 3

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Problem 1 *Show that left adjoints preserve initial objects: that is, if*

$$\mathcal{A} \underset{G}{\overset{F}{\rightleftharpoons}} \mathcal{B}$$

and I is an initial object of $\mathcal{A}$, then $F(I)$ is an initial object of $\mathcal{B}$. Dually, show that right adjoints preserve terminal objects.

Problem 2 *Let G be a group.*

(a) What interesting functors are there (in either direction) between Set and the category $[G, \mathbf{Set}]$ of left G-sets? Which of those functors are adjoint to which?

(b) Similarly, what interesting functors are there between $\mathbf{Vect}_k$ and the category $[G, \mathbf{Vect}_k]$ of k-linear representations of G, and what adjunctions are there between those functors?

Problem 3 *Fix a topological space X, and write $\mathcal{O}(X)$ for the poset of open subsets of X, ordered by inclusion. Let*

$$\Delta : \mathbf{Set} \to [\mathcal{O}(X)^{op}, \mathbf{Set}]$$

be the functor assigning to a set A the presheaf ΔA with constant value A (for the definition of presheaves *see Definition 1.2.15). Exhibit a chain of adjoint functors*

$$\Lambda \dashv \Pi \dashv \Delta \dashv \Gamma \dashv \nabla.$$

Problem 4 *Let $(F, G, \eta, \varepsilon)$ be an equivalence of categories, as in Definition 1.3.15. Prove that F is left adjoint to G (heeding the warning in Remark 2.2.8).*

Problem 5 *Let $\mathcal{A} \underset{U}{\overset{F}{\rightleftharpoons}} \mathbf{Set}$ be an adjunction. Suppose that for at least one $A \in \mathcal{A}$, the set $U(A)$ has at least two elements. Prove that for each set S, the unit map $\eta_S : S \to UF(S)$ is injective. What does this mean in the case of the usual adjunction between* **Grp** *and* **Set***?*

Problem 6 *(a) Let* $\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$ *be an adjunction with unit* η *and counit* ε*. Write Fix(GF) for the full subcategory of* $\mathcal{A}$ *whose objects are those* $A \in \mathcal{A}$ *such that* η_A *is an isomorphism, and dually Fix(FG)* $\subseteq \mathcal{B}$*. Prove that the adjunction* $(F, G, \eta, \varepsilon)$ *restricts to an equivalence* $(F', G', \eta', \varepsilon')$ *between Fix(GF) and Fix(FG).*

(b) Part (a) shows that every adjunction restricts to an equivalence between full subcategories in a canonical way. Take some examples of adjunctions and work out what this equivalence is.

Problem 7 *(a) Show that for any adjunction, the right adjoint is full and faithful if and only if the counit is an isomorphism.*

(b) An adjunction satisfying the equivalent conditions of part 1 is called a reflection. (Compare Example 2.1.3(d).) Of the examples of adjunctions given in this chapter, which are reflections?

Problem 8 *(a) Let* $f : K \to L$ *be a map of sets, and denote by* $f^* : \mathcal{P}(L) \to \mathcal{P}(K)$ *the map sending a subset* S *of* L *to its inverse image* $f^{-1}S \subseteq K$*. Then* f^* *is order-preserving with respect to the inclusion orderings on* $\mathcal{P}(K)$ *and* $\mathcal{P}(L)$*, and so can be seen as a functor. Find left and right adjoints to* f^**.*

(b) Now let X *and* Y *be sets, and write* $p : X \times Y \to X$ *for first projection. Regard a subset* S *of* X *as a predicate* $S(x)$ *in one variable* $x \in X$*, and similarly a subset* R *of* $X \times Y$ *as a predicate* $R(x, y)$ *in two variables. What, in terms of predicates, are the left and right adjoints to* p^**? For each of the adjunctions, interpret the unit and counit as logical implications. (Hint: the left adjoint to* p^* *is often written as* $\exists_Y$*, and the right adjoint as* $\forall_Y$*.)*

Problem 9 *Given a functor* $F : \mathcal{A} \to \mathcal{B}$ *and a category* $\mathcal{S}$*, there is a functor* $F^* : [\mathcal{B}, \mathcal{S}] \to [\mathcal{A}, \mathcal{S}]$ *defined on objects* $Y \in [\mathcal{B}, \mathcal{S}]$ *by* $F^*(Y) = Y \circ F$ *and on maps* α *by* $F^*(\alpha) = \alpha F$*. Show that any adjunction* $\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$ *and category* $\mathcal{S}$ *give rise to an adjunction*

$$[\mathcal{A}, \mathcal{S}] \underset{F^*}{\overset{G^*}{\rightleftarrows}} [\mathcal{B}, \mathcal{S}].$$

(Hint: use Theorem 2.2.5.)

Problem 10 *Let* p *be a prime number. Show that the functor* U_p*:* $\mathbf{Grp} \to \mathbf{Set}$ *defined in Example 4.1.5 is isomorphic to* $\mathbf{Grp}(Z/pZ, -)$*. (To check that there is an isomorphism of functors – that is, a natural isomorphism – you will first need to define* U_p *on maps. There is only one sensible way to do this.)*

Problem 11 *Using the result of Exercise 0.13(a), prove that the forgetful functor*

$$\mathbf{CRing} \to \mathbf{Set}$$

is isomorphic to $\mathbf{CRing}(Z[x], -)$*, as in Example 4.1.14.*

Problem 12 *The **Sierpinski space** is the two-point topological space S in which one of the singleton subsets is open but the other is not. Prove that for any topological space X, there is a canonical bijection between the open subsets of X and the continuous maps $X \to S$. Use this to show that the functor $\mathcal{O}$:* $\mathbf{Top} \to \mathbf{Set}$ *of Example 4.1.19 is represented by S.*

Problem 13 *Let M:* $\mathbf{Cat} \to \mathbf{Set}$ *be the functor that sends a small category $\mathcal{C}$ to the set of all maps in $\mathcal{C}$. Prove that M is representable.*

Problem 14 *Let $\mathcal{A}$ be a locally small category. Prove each of the following statements directly (without using the Yoneda lemma).*

(a) $H_\bullet : \mathcal{A} \to [\mathcal{A}^{op}, \mathbf{Set}]$ *is faithful.*

(b) $H_\bullet$ *is full.*

(c) Given $A \in \mathcal{A}$ and a presheaf X on $\mathcal{A}$, if $X(A)$ has an element u that is universal in the sense of Corollary 4.3.2, then $X \cong H_A$.

Problem 15 *Let $\mathcal{B}$ be a category and $J : G' \to \mathcal{D}$ a functor. There is an induced functor*

$$J \circ - : [\mathcal{B}, G] \to [\mathcal{B}, \mathcal{D}]$$

defined by composition with J.

(a) Show that if J is full and faithful then so is $J \circ -$.

(b) Deduce that if J is full and faithful and $G, G' : \mathcal{B} \to G'$ with $J \circ G \cong J \circ G'$ then $G \cong G'$.

(c) Now deduce that right adjoints are unique: if $F : \mathcal{A} \to \mathcal{B}$ and $G, G' : \mathcal{B} \to \mathcal{A}$ with $F + G$ and $F + G'$ then $G \cong G'$. (Hint: the Yoneda embedding is full and faithful.)